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A Tree-Structured Deterministic Self-Similar Network

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Abstract. *Self-similarity is a property existing in natural, social, technological, biological, and many other networks. The whole and the parts, or one part and another part of self-similar networks present some similarity. In the past dozen years, researchers have proposed some deterministic self-similar networks by utilizing various kinds of mechanisms. In this paper, we propose a new deterministic self-similar network model based on the tree fractal. Furthermore, we give the analytic solutions to several important topological characteristics of the proposed model.*
Keywords: Deterministic models, Tree-structured network, Self-similar network, Degree distribution, Average path length, Clustering coefficient.

1. **Introduction.** Complex systems widely exist in natural, social, biological, engineering, and many other fields. It is one of the most challenging issues to research the modeling problem of complex systems, and it has an important significance for deeply analyzing and scientifically understanding the internal relationship between system structures, functions and dynamics. The complex network theory provides a new way of thinking and perspective for the research of complex science, especially complex systems.

Complex network models can be divided into probabilistic models and deterministic models. The famous probabilistic models include ER random network [1], WS small-world network [2], NW small-world network [3] and BA scale-free network [4] and so on. Although probabilistic models conform to the main features of most of real-life networks, it is difficult for people to have an intuitive understanding of the formation of complex networks and the interaction between different nodes. On the other hand, if a real-life network (e.g., a neural network) has fixed interconnections, the probabilistic technique is no longer suitable to model its generation process. Constructing network models in deterministic manners not only has important theoretical significance, but also has potential application value. The main advantage of deterministic networks is that their topological features can be computed analytically.

Deterministic models can be divided into models with self-similarity and without self-similarity. Self-similarity refers to that there is similarity between the overall and the parts of a system, or between one section and another section. Many researchers have proposed deterministic models without self-similarity. Comellas et al. [5] proposed the first deterministic small-world network based on graph-theoretic methods. Zhang et al. [6] presented a deterministic small-world network created by edge iterations. Lu and Guo [7] introduced a deterministic small-world network derived from the deterministic

uniform recursive tree. Guo et al. [8] constructed a tree-structured deterministic small-world network. In addition, Lu et al. [9] presented a deterministic scale-free small-world network model. Other researchers have also proposed deterministic models with self-similarity. Dorogovtsev et al. [10] proposed a pseudofractal scale-free graph which grows under the mechanism of preferential linking nodes with higher numbers of links attach higher numbers of new edges. Zhang et al. [11] presented an incompatibility network based on Sierpinski fractals. In addition, Hinczewski and Berker [12] constructed a scale-free hierarchical-lattice small-world network. Inspired by forked branches in the nature, in this paper, we propose a simple deterministic self-similar network.

2. **Proposed Deterministic Self-similar Network.** For a common tree in real-life, its whole and any branch are very similar in form, and the only difference lies in their different sizes. We construct our self-similar network based on the tree fractal. Assume the obtained network after t iterations is SS_t that has N_t nodes and E_t edges, where $t = 0, 1, 2, ..., T-1$, and T is the number of iterations performed. Assume each node is labeled with a natural number increasing with the generation time, then the proposed generation process can be illustrated as follows:

Step 0: Initialization. Set $t = 0$, SS_0 contains an edge that connects two nodes labeled as '1' and '2'. Obviously, $N_0 = 2$ and $E_0 = 1$.

Step 1: Generation of SS_1 from SS_0. A new node labeled as '3' is generated at the midpoint of the edge generated in Step 0. And two nodes labeled as '4' and '5' branch from Node '3'. Thus, $N_1 = 5$ and $E_1 = 4$.

Step 2: Generation of SS_{t+1} from SS_t for $t > 0$. A new node is generated at the midpoint of the each newly-generated edge in SS_t. And two nodes branch from each new node. Obviously, after the above step, we have $N_{t+1} = 2N_t + 1$ and $E_{t+1} = 2N_t$.

Step 3: If $t < T - 1$, set $t = t + 1$ and go to Step 2. Otherwise, the algorithm is terminated.

The above iterative process is repeated for $T - 1$ times, and then we can obtain a deterministic self-similar network as shown below. In fact, in the above generation process, we mimic the growth of a tree to some extent. Fig. 1 shows the obtained network after the first four iterations. According to the relationships $N_{t+1} = 2N_t + 1$ and $E_{t+1} = 2N_t$ together with the initial conditions $N_0 = 2$ and $E_0 = 1$, we can easily prove that $N_t = 3 \times 2^t - 1$ and $E_t = 3 \times 2^t - 2$, thus we can obtain the average node degree as follows

$$< k >_t = \frac{2E_t}{N_t} = \frac{2 \times (3 \times 2^t - 2)}{3 \times 2^t - 1} = 2 \times (1 - \frac{1}{3 \times 2^t - 1}) \tag{1}$$

Since $\lim_{t \to \infty} < k >_t = 2$, we can see that the proposed network is a sparse graph whose nodes have fewer links than possible.

3. **Topological Properties.**

3.1. **Degree Distribution and Degree Correlation.** Degree distribution is one of the most important topological characters of a network. The degree of Node i is defined as the number of edges it connects to other nodes, and degree distribution $P(k)$ is defined as the fraction of nodes in the network with degree k. According to the iteration algorithm, we can easily prove that the possible degree values in the proposed network are 1 and 4. Thus, we can easily obtain

$$P(k) = \frac{2^{t+1}}{3 \times 2^t - 1}\delta(k-1) + \frac{2^t - 1}{3 \times 2^t - 1}\delta(k-4) \tag{2}$$

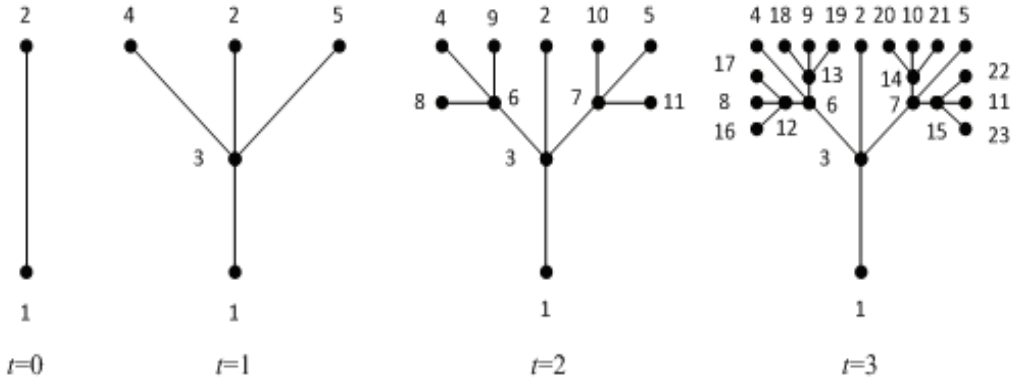

FIGURE 1. The first four iterations of the growth of the proposed network

Where $\delta(k) = 1$ for $k = 0$ and $\delta(k) = 0$ for $k \neq 0$. When $t \to \infty$, we can easily obtain the following degree distribution

$$\lim_{t\to\infty} P(k) = \frac{2}{3}\delta(k-1) + \frac{1}{3}\delta(k-4) \tag{3}$$

Therefore, the degree distribution of the proposed network is discrete.

Degree correlation describes the relationship between nodes with high-degree and nodes with low-degree in a network. The Pearson correlation coefficient r [13] can be used to describe degree correlation. It is defined as follows

$$r = \frac{M^{-1}\sum_i j_i k_i - [M^{-1}\sum_i \frac{1}{2}(j_i + k_i)]^2}{M^{-1}\sum_i \frac{1}{2}(j_i^2 + k_i^2) - [M^{-1}\sum_i \frac{1}{2}(j_i + k_i)]^2} \tag{4}$$

Where M is the number of edges and j_i, k_i are the degrees of the nodes at the ends of the i-th edge, with $i = 1, 2, ..., M$, r lies in the range $-1 \leq r \leq 1$. When $r < 0$, the network is disassortative. When $r > 0$, the network is assortative. We give the simulation results of r in Fig. 2. From Fig. 2, we can see that the Pearson correlation coefficient r of the proposed network is less than zero. So it is disassortative.

3.2. **Clustering Coefficient.** The clustering coefficient is a measure of degree to which nodes in a graph tend to cluster together. The clustering coefficient of a network is the average of the local clustering coefficients over all the nodes in the network. The local clustering coefficient C_i for Node i with degree k_i is defined as the number of links n_i that actually exist between its nearest neighbors divided by the number of links that could possibly exist between them, i.e., $C_i = 2n_i/[k_i(k_i - 1)]$. Since there are no triangles in the proposed network, we can easily know that the clustering coefficient of it is zero. The network is not a kind of small-world network.

3.3. **Diameter and Average Path Length.** Average path length (APL) is also an important parameter to characterize a network. APL is defined as the average number of edges along the shortest paths for all possible pairs of nodes in a connected network. It is difficult to obtain the analytic solution of APL for most network models. We can adopt another parameter called diameter to analyze APL indirectly. Diameter is defined as the maximal distance between any two nodes in the network, which characterizes the maximum communication delay in the network. If a network is with a small diameter, then this network is undoubtedly with a short APL. Here, we denote the diameter at Iteration t as $D(t)$. Because of the tree structure, one can easily see that the diameter

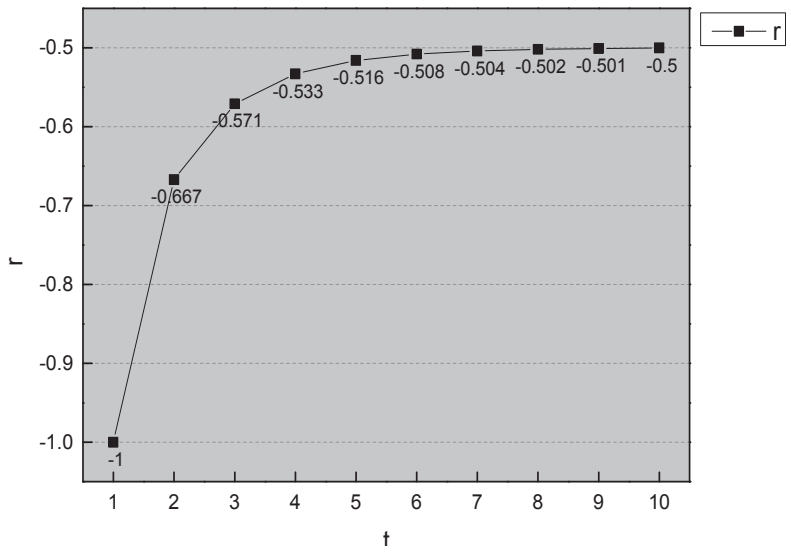

FIGURE 2. The r versus the number of the iterations

always lies between one top node from left branch and one top node from right branch at this iteration. Take the node pair $(2^{t+1}, 3 \times 2^t - 1)$ for example, while traveling from Node 2^{t+1} to Node $3 \times 2^t - 1$, the shortest path includes $2t$ edges. Thus, the diameter for the proposed network has the following simple formula

$$D(t) = 2t = \frac{2\ln(N_t + 1)}{\ln 2} - \frac{2\ln 3}{\ln 2} \tag{5}$$

Thus, the diameter D grows logarithmically with the number of nodes. Because the average path length is smaller than D, the APL should increase more slowly. To show the relationship more clearly, we give the simulation results in Fig. 3. To have an insight into the frequency distribution of shortest path lengths, we also give a plot in Fig. 4 to show the fraction of cases in which the shortest path between a pair of nodes is m, where $m = 1, 2, ..., D(t)$ for $t = 4$. From Fig. 4, we can see that for $t = 4$, most shortest paths are with lengths 5 and 6, which corresponds to the average path length 5.02 as shown in Fig. 3.

According to the above discussion, we can conclude that our model is a deterministic self-similar network, for it is based on the tree fractal and has self-similarity. The network is sparse with a short average path length.

4. **Conclusions.** In this paper, we have presented a deterministic self-similar model based on the tree fractal. We have derived the analytic solutions for degree distribution, clustering coefficient and diameter of the deterministic model. The network satisfies the two necessary properties for small-world networks, but its clustering coefficient is zero. So it is not a small-world network. The proposed model provides a new way of thinking to generate a self-similar network with specific properties. Our future work will concentrate on how to add extra links to improve the clustering coefficient from 0 to a high value. This will help us to obtain a small-world network.

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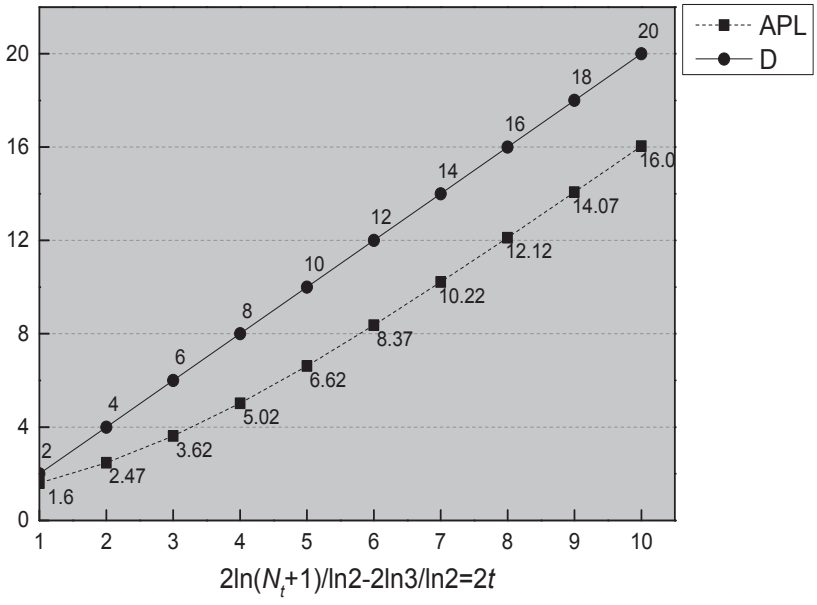

FIGURE 3. The APL and D versus the logarithm of the number of nodes

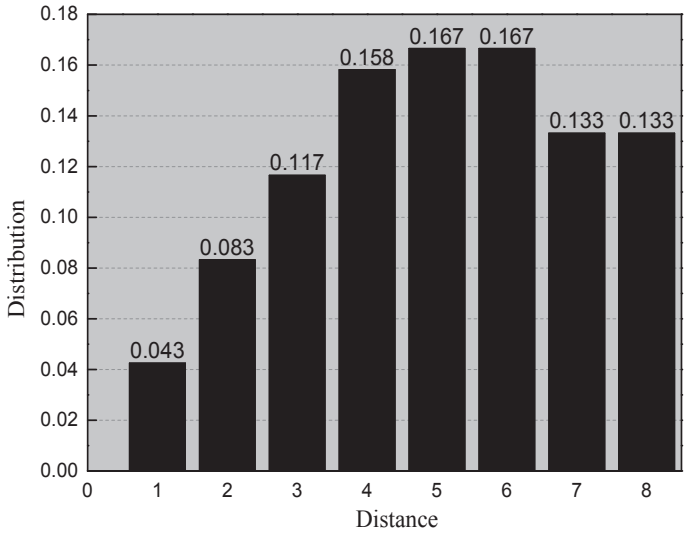

FIGURE 4. The frequency distribution of shortest path lengths for t=4

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