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Split Bregman Iteration Algorithm for An Adaptive Hybrid Variation Model Based Image Restoration

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Abstract. *In this paper, instead of solving the Lagrange equations directly, we introduce a new unconstrained problem by applying operator splitting and penalty techniques to take replace of the original minimizing issue. We have proved Split Bregman method is efficient to restore the degraded images.*
Keywords: Image restoration; Split Bregman; The Euler-Lagrange equation; Hybrid variation.

1. **Introduction.** In recent years, image restoration technology research and application mainly concentrated in areas, such as space exploration, astronomical observation, material research, remote sensing, military science, biological sciences, medical imaging, traffic monitoring, criminal investigation fields. In the image acquisition, transmission and storage in the process, due to various factors, such as the effect of atmospheric turbulence, camera equipment of optical system in the diffraction, nonlinear characteristics of the sensor, the aberrations of the optical system, imaging devices and objects between the relative motion, will inevitably cause image degraded. Therefore, image restoration is necessary to suppress noise and improve image quality. The process of image restoration is that, an observed image u_0 is divided into an original image u and an additive noise n ,

$$u_0 = Au + n, \tag{1}$$

where A is a bounded linear operator, n is the white Gaussian noise. For degraded image u_0 , any small perturbation may result in the solution $A^{-1}u_0$ to be far away from the original image u, many different approaches have been proposed for image denoising. Among these methods, the total variation(TV) model is distinguished for excellent edge preserving ability[1, 2], it became one of the most widely used regularizers in image restoration. The energy functional[3] is of the following form:

$$E(u) = \int_\Omega |\nabla u| dxdy + \frac{\lambda}{2} \int_\Omega (u - u_0)^2 dxdy. \tag{2}$$

Where u represents the pixel value of the image, and it meets the following noise constraints:

$$\frac{1}{|\Omega|}\int_{\Omega}(u-u_0)^2dxdy=\sigma^2, \tag{3}$$

the original image will be contaminated by poor white noise. The pixel value of the corresponding pixels of the original image is adopted by the Lagrange multiplier method, the Euler-Lagrange equation of (2) is that

$$-\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)+\lambda(u-u_0)=0. \tag{4}$$

Where ∇ is the gradient operator. λ is the regularization parameter which balances the regularization term and the data fidelity term. In order to solve the neighborhood information of the image in the flat area, L_2 model introduced by Tikhonov and Arsenin, is given by

$$E(u)=\int_{\Omega}|\nabla u|^2dxdy+\frac{\lambda}{2}\int_{\Omega}(u-u_0)^2dxdy. \tag{5}$$

The corresponding Euler-Lagrange equation is

$$\Delta u+\lambda(u-u_0)=0. \tag{6}$$

From the above discussion, we can see that the L_1 norm variation model can effective preserve the edges and corners . However, it yields unwanted stair-casing [4,5]. While the L_2 norm variation model can be realized isotropic diffusion in the flat region, the image edge preserving ability is very poor. The most ideal method is that, it not only can protect the edge information, but also can make the flat area is well spread. Thus this paper will combine two above models organically, the proposed hybrid model is as follows:

$$E(u)=g\int_{\Omega}|\nabla u|dxdy+\frac{1-g}{2}\int_{\Omega}|\nabla u|^2dxdy+\frac{\lambda}{2}\int_{\Omega}(u-u_0)^2dxdy. \tag{7}$$

Where g(x,y) is a stopping function chosen as

$$g(x,y)=\begin{cases}0 & if \quad |\nabla G_\sigma * u_0|\le\beta_1,\\ 1-\frac{1}{1+10*|\nabla G_\sigma * u_0(x,y)|^2} & if \quad \beta_1\le|\nabla G_\sigma * u_0|\le\beta_2,\\ 1 & if \quad |\nabla G_\sigma * u_0|\ge\beta_2.\end{cases}$$

where $\lambda\ge 0$ is the coefficient of the fidelity term, G_σ is the Gaussian kernel and σ is the standard deviation. Firstly, when $g=1$, this pixel is located in a very prominent edge section, then the edge need to recover. At this time, the proposed hybrid model becomes the L_1 norm variation model, which is able to achieve well for preserving edges. Secondly, when $g=0$, this pixel is located in the flat area of the image, at this time, the proposed hybrid model becomes L_2 norm variation model, which can eliminate the staircase effect. It can be seen that the parameter g is able to control the diffusion coefficient. From the above analysis, by adding the parameter g , the proposed hybrid model[6,7] can adaptively choice the L_1 norm variation model or L_2 norm variation model according to the location of the pixels point. the proposed hybrid model performs very well for removing noise while preserving edges.

The rest of the paper is organized as follows. In section 2, we describe the necessary definitions and preliminaries about the proposed model. In Section 3, we give a brief review of some related iterative algorithms. In section 4, we apply two algorithms to compute the hybrid model. In section 5, numerical experiments with two algorithms are presented. Finally, a brief summary is given in section 6.

2. **Preliminaries.** In this section, we give some definitions,propositions and prove theorem.

Definition 1. Let $\Omega \subseteq R^N$ be a bounded open domain, $u \in L^1_{loc}(\Omega)$,then the total variational of u is defined by

$$\int_\Omega |Du|dxdy = sup\left\{\int_\Omega udiv\omega dxdy : \omega = (\omega_1, \omega_2) \in C_0^\infty(\Omega), |\omega| \le 1\right\}.$$

Proposition 1.(Lower semicontinuous). Suppose that $\{u_i\}_{i=1}^\infty$ and $u^* \in L^1(\Omega)$ is such that $u_i \to u^*$ in $L^1(\Omega)$, then$\int_\Omega |\nabla u^*| \le \liminf \int_\Omega |\nabla u_i|$. Where $BV(\Omega)$ is a Banach space[8].

Proposition 2. Let $u^* \in L^2(\Omega) \cap BV(\Omega)$, then there exists a minimizing sequence $\{u_i\}_{i=1}^\infty \subset BV(\Omega)$ such that $\lim\limits_{i\to\infty} |u_i - u^*| = 0$ and $\lim\limits_{i\to\infty} \int_\Omega |\nabla u_i| = \int_\Omega |\nabla u|$. we can prove the existence and uniqueness of the optimization problem (7) solution.

Theorem 2.1. *The problem (7) has a unique solution in $L_2(\Omega) \cap BV(\Omega)$.*

Proof. Let $\{u_i\}_{i=1}^\infty$ is a minimizing sequence. By the Kondrachov compactness theorem, the sequence $\{u_i\}_{i=1}^\infty$ is precompact in $L_2(\Omega) \cap BV(\Omega)$. Namely, there exists a function u^* satisfying $u_i \to u^*$ a.e. Since the function is convex and coercive in $L_2(\Omega) \cap BV(\Omega)$, the lower semicontinuity property is satisfied, then we have

$$\lim_{i\to\infty} \inf ||u_i||_{BV} \ge ||u^*||_{BV}, \qquad ||u_i||_{L_2(\Omega)} \ge ||u^*||_{L_2(\Omega)},$$

and

$$\inf\left\{g\int_\Omega |\nabla u|dxdy + \frac{1-g}{2}\int_\Omega |\nabla u|^2 dxdy + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 dxdy\right\}$$

$$\ge \lim_{i\to\infty}\inf\left\{g\int_\Omega |\nabla u_i|dxdy + \frac{1-g}{2}\int_\Omega |\nabla u_i|^2 dxdy + \frac{\lambda}{2}\int_\Omega (u_i-u_0)^2 dxdy\right\}$$

$$\ge \left\{g\int_\Omega |\nabla u^*|dxdy + \frac{1-g}{2}\int_\Omega |\nabla u^*|^2 dxdy + \frac{\lambda}{2}\int_\Omega (u^*-u_0)^2 dxdy\right\}.$$

Which concludes that u^* is minimum point of (7). Next, we give the proof of the uniqueness. Let u^* and v^* are two minimum point of (7). following from the convexity, we easily obtain that $\nabla u^* = \nabla v^*$, which means that $u^* = v^* + c$, Additionally, considering that $F(u) = \frac{1-g}{2}||\nabla u||^2 + \frac{\lambda}{2}||u-u_0||_2^2$ is strictly convex, we conclude that$Au^* = Av^*$, therefore $Ac = 0$. Note that A is linear and the functional (7) is coercive in $L^2(\Omega) \cap BV(\Omega)$, , we deduce that $c = 0$ and $u^* = v^*$.

3. **Bregman-Related Algorithms.** Recently, split Bregman iteration has attracted extensive attention in the field of signal recovery. This method is valid for norm minimization problems. The essential idea is to convert a constrained optimization problem into an unconstrained one. In this section, We will introduce the calculation methods in great detail to solve this kind of problem.

3.1. **Bregman iteration.** As we have mentioned above, Bregman iteration was originally introduced and studied in image processing by Osher et al. in [9], their core idea is to transform a constrained optimization problem into a series of unconstrained problems. Then through the Bregman distance to perform variable separation iteration quickly. Two convex energy functions are $E(u)$ and$H(u)$, the minimization problem is

$$\min_u J(u) + H(u). \tag{8}$$

The Bregman iteration was originated from the concept of distance [10]

$$E_J^P(u, u^k) = J(u) - J(u^k) - (P, u - u^k). \tag{9}$$

Then the Bregman iteration for (8) is

$$u^{k+1} = \min_u E_J^P(u, u^k) + H(u), \tag{10}$$

$$u^{k+1} = \min_u J(u) - J(u^k) - (P, u - u^k) + H(u). \tag{11}$$

In order that(11) is well defined for $k+1$, we have:

$$P^{k+1} = P^k - \nabla H(u^{k+1}). \tag{12}$$

When $H(u) = \frac{\lambda}{2}||Au - f + p^k||_2^2$ and A is linear, the Bregman iteration of (11) and (12) is equivalent to the following simplified version [11]:

$$u^{k+1} = \min_u J(u) + \frac{\lambda}{2}||Au - f + p^k||_2^2, \tag{13}$$

$$P^{k+1} = P^k + Au^{k+1} - f. \tag{14}$$

3.2. **Split Bregman iteration.** We consider the general L_1 norm problem, assume that $J(u) = |d|$ and $d = \tau(u)$. Then (8) become to the following form:

$$\min_{u,d} ||d||_1 + H(u), \tag{15}$$

such that $d = \Phi(u)$.

Here, assume that $H(u)$ and$|\tau(u)|$ are convex function, and $\tau(u)$ is differentiable. In order to solve this problem [12,13], we transform it into an unconstrained problem:

$$\min_{u,d} ||d||_1 + H(u) + \frac{\mu}{2}||d - \tau(u)||_2^2. \tag{16}$$

Then the problem (16) can be solved with the simplified two-phase iterative algorithm, we get

$$(u^{k+1}, d^{k+1}) = \min_{u,d} ||d||_1 + H(u) + \frac{\mu}{2}||d - \tau(u) - b^k||_2^2, \tag{17}$$

$$b^{k+1} = b^k + \left(\tau(u^{k+1}) - d^{k+1}\right). \tag{18}$$

(17) and (18) are the two iteration formula of the split Bregman iteration [14,15,16]. In order to compute (17), we must minimize the subproblems of u and d as follows:

$$(u^{k+1}, d^{k+1}) = \min_{u,d} ||d||_1 + H(u) + \frac{\mu}{2}||d - \tau(u) - b^k||_2^2, \tag{19}$$

$$d^{k+1} = \min_{u,d} ||d||_1 + \frac{\mu}{2}||d^k - \tau(u^{k+1}) - b^k||_2^2. \tag{20}$$

We compute d by using soft shrinkage operators :

$$u_j^{k+1} = shrink\left(\tau(u)_j + b_j^k, \frac{1}{\mu}\right), \tag{21}$$

where $shrink(x,y) = \frac{x}{|x|} * \max(|x| - y, 0)$.

This algorithm is extremely fast and simple, it is widely uded in image processing and the convergence analysis were given in Refs.

4. **Computational method.**

4.1. **Algorithm 1: the variational method.**

Numerically, we use the following minimization problem to approximate (7):

$$\min_u g \int_\Omega |\nabla u| dxdy + \frac{1-g}{2} \int_\Omega |\nabla u|^2 dxdy + \frac{\lambda}{2} \int_\Omega (u - u_0)^2 dxdy. \tag{22}$$

Next, we will derive the Euler-Lagrange equation of energy (22), for any real number ε and ν functions ,

$$\begin{aligned} \frac{dE(u+\varepsilon\nu)}{d\varepsilon} &= \frac{d}{d\varepsilon} \int_\Omega g|\nabla u dxdy + \varepsilon \nabla \nu| dxdy + \frac{1-g}{2} |\nabla u + \varepsilon \nabla \nu|^2 dxdy \\ &\quad + \frac{\lambda}{2} |u + \varepsilon\nu - u_0|^2 dxdy \end{aligned}$$

when $\varepsilon = 0$, we will obtain

$$\frac{dE(u+\varepsilon\nu)}{d\varepsilon}|_{\varepsilon=0} = \int_\Omega \frac{g}{|\nabla u|} \nabla\nu \cdot \nabla u + (1-g)\nabla u \cdot \nabla \nu + \lambda(u - u_0) dxdy = 0. \tag{23}$$

In conclusion, we get the Euler-Lagrange equation as follows:

$$\begin{cases} -\nabla \cdot \left(\frac{g}{|\nabla u|} \nabla u \right) - \nabla \cdot ((1-g) \cdot \nabla u) + \lambda (u - u_0) = 0, \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{24}$$

4.2. **Algorithm 2: Split Bregman method.**

In this section, we apply the Split-Bregman method to solve the proposed model (7). By introducing several auxiliary variables $d = (d_x, d_y)$, make the following substitutions for model (22): $d_x = \nabla_x u, d_y = \nabla_y u$. This yields a constrained optimization problem:

$$\min_u \left\{ \frac{\lambda}{2} ||u - u_0||_2^2 + \frac{1-g}{2} ||\nabla u||_2^2 + g||(d_x, d_y)||_1 \right\}, \tag{25}$$

s.t. $\quad d_x = \nabla_x u, d_y = \nabla_y u.$

Using 2-norm to enforce the above constraints, it becomes:

$$\begin{aligned} \min_{u,d_x,d_y} \quad & \frac{\lambda}{2} ||u - u_0||_2^2 + \frac{1-g}{2} ||\nabla u||_2^2 + g||(d_x, d_y)||_1 \\ & + \frac{\sigma}{2} ||d_x - \nabla_x u||_2^2 + \frac{\sigma}{2} ||d_y - \nabla_y u||_2^2. \end{aligned} \tag{26}$$

Applying the Split-Bregman iteration to solve the constraints problem, we can get the following iterative scheme (27) and (28):

$$\begin{aligned} (u^{k+1}, d_x^{k+1}, d_y^{k+1}) &= \arg \min_{u,d_x,d_y} \frac{\lambda}{2} ||u - u_0||_2^2 + \frac{1-g}{2} ||\nabla u||_2^2 + g||(d_x, d_y)||_1 \\ &\quad + \frac{\sigma}{2} ||d_x - \nabla_x u||_2^2 + \frac{\sigma}{2} ||d_y - \nabla_y u||_2^2, \end{aligned} \tag{27}$$

$$b_x^{k+1} = b_x^k + (\nabla_x u^{k+1} - d_x^{k+1}), \qquad b_y^{k+1} = b_y^k + (\nabla_y u^{k+1} - d_y^{k+1}). \tag{28}$$

where the parameters $\sigma > 0$, k is the iteration times. The formula (27) can be solved effectively by minimizing with respect to u and (d_x, d_y). Hence we need to solve the following two minimization subproblems:

$$u^{k+1} = \arg\min_{u} \quad \frac{\lambda}{2}||u-u_0||_2^2 + \frac{1-g}{2}||\nabla u||_2^2 + \frac{\sigma}{2}||d_x - \nabla_x u - b_x^k||_2^2 + \frac{\sigma}{2}||d_y - \nabla_y u - b_y^k||_2^2, \tag{29}$$

$$(d_x^{k+1}, d_y^{k+1}) = \arg\min_{d_x, d_y} g||(d_x, d_y)||_1 + \frac{\sigma}{2}||d_x - \nabla_x u - b_x^k||_2^2 + \frac{\sigma}{2}||d_y - \nabla_y u - b_y^k||_2^2. \tag{30}$$

1.Since the right side of (29) is differentiable, for solving the u-subproblem , we derive the optimality condition

$$[\lambda - (1-g+\sigma)\Delta]u - \lambda u_0 - \sigma\nabla_x^\top(d_x^k - b_x^k) - \sigma\nabla_y^\top(d_y^k - b_y^k) = 0, \tag{31}$$

where $\Delta = -(\nabla_x^\top\nabla_x + \nabla_y^\top\nabla_y)$, $\nabla_x^\top$ and$\nabla_y^\top$ are the adjoint matrix of $\nabla_x$and ∇_y respectively. For reducing the complexity of the algorithm, we apply the fast Fourier transform (FFT) [17, 18] to solve the u-subproblem

$$u^{k+1} = F^{-1}\left[\frac{F\left(\lambda u_0 I + \sigma\nabla_x^\top(d_x^k - b_x^k) + \sigma\nabla_y^\top(d_y^k - b_y^k)\right)}{F\left(\lambda - (1-g+\sigma)\Delta\right)}\right]. \tag{32}$$

2.The d-subproblem can be solved by shrinkage formulation [19], the closed-form solution is:

$$d_x^{k+1} = \max\left(s^k - \frac{g}{\sigma}, 0\right)\cdot\frac{s_x^k}{s^k}, \tag{33}$$

$$d_y^{k+1} = \max\left(s^k - \frac{g}{\sigma}, 0\right)\cdot\frac{s_y^k}{s^k}. \tag{34}$$

Where

$$s_x^k = \nabla_x u^{k+1} + b_x^k, \quad s_y^k = \nabla_y u^{k+1} + b_y^k, \quad s^k = \sqrt{(s_x^k)^2 + (s_y^k)^2}.$$

We summarize the algorithm as follows:

1.Initialization: set$u^0 = u_0$,and $d_x^0 = d_y^0 = b_x^0 = b_y^0 = 0$.
2.For$k = 0, 1, \cdots$, do.

- a: solve (32) to get u^{k+1} ,
- b: solve (33) and (34) to get d_x^{k+1} and d_y^{k+1} respectively,
- c: update b_x^{k+1} and b_y^{k+1} by (28).

End do still some stooping rule meet.

5. **Numerical Experiments.** In this section, several experiments are performed to demonstrate the effectiveness of our proposed split Bregman iteration algorithm and the adaptive hybrid variation model. All experiments are generated in MATLAB 7.8 environment on a desktop with Windows XP operating system; 2.0 GHz Intel core 2DUO CPU, and 2GB memory. The performance of all algorithms is measured by the peak signal to noise ratio (PSNR) and mean squared error MSE.

$$PSNR = 10\lg\frac{255\times 255}{\frac{1}{M\times N}\sum_{i,j}(u'_{i,j} - u_{i,j})^2},$$

Table 1. Comparison of the recovered results

Figure	Method	Niter	Time(s)	PSNR	MSE
Lena	TV	165	61.949299	41.0489	5.0969
Lena	SBI	180	3.925539	42.0802	4.0401

Table 2. Comparison of the recovered results

Figure	Method	Niter	Time(s)	PSNR	MSE
Cameraman	TV	174	60.630745	44.1710	3.4853
Cameraman	SBI	186	3.196993	44.9083	2.0949

$$MSE = \frac{1}{M \times N} \sum_{i,j} (u'_{i,j} - u_{i,j})^2,$$

If there is a higher peak signal to noise ratio and a lower mean square error, the image quality will be better. Moreover, the stopping criterion of all algorithms must satisfies the following inequality:

$$\frac{||u^{k+1} - u^k}{||u^{k+1}||} \leq 5 \times 10^{-4}.$$

Three classical gray scale images are used for synthetic degradations in our experiments. For this adaptive hybrid variation model, we still use the traditional fixed point iteration method and split Bregman iteration method. We choose $\lambda = 0.8$ and $g = 0.5$,$\beta_1 = 50$,$\beta_2 = 150$ in all experiments. The stopping condition of the iterations in all experiments is $||u^{k+1} - u^k||/||u^{k+1}|| \leq 5 \times 10^{-4}$. The parameters in split Bregman iteration method choose $\lambda = 2e + 6$ and $g = 0.4$, $\sigma = 1e - 4$. Table [1, 2, 3] list data to compare three experiments based on image restoration. As we can seen from the data in table [1, 2, 3] ,split Bregman iteration algorithm not only reach stopping standards significantly higher than TV, but also restoring time is significantly less than TV, which means that split Bregman iteration algorithm has faster convergence speed. table [1, 2, 3] show that split Bregman algorithm is better than the TV algorithm.

Figure 1. Recovered results. (a) degraded image, (b) TV , (c) Split-Bregman.

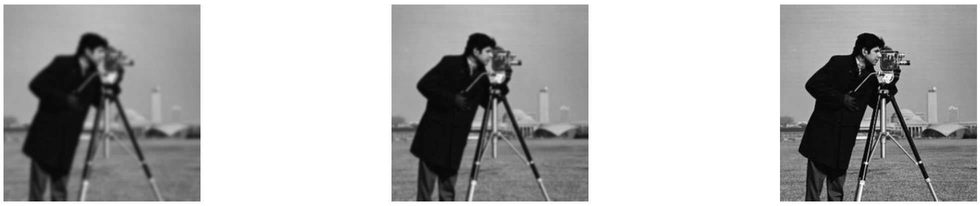

FIGURE 2. Recovered results. (a) degraded image, (b) TV , (c) Split-Bregman.

FIGURE 3. Recovered results. (a) degraded image, (b) TV , (c) Split-Bregman.

TABLE 3. Comparison of the recovered results

Figure	Method	Niter	Time(s)	PSNR	MSE
Boat	TV	165	61.949299	41.0489	5.0969
Boat	SBI	180	3.925539	42.0802	4.0401

6. **Conclusions.** In this paper, we propose an adaptive hybrid variation model and applied the split Bregman iteration algorithm to solve it. Several numerical experiments show that split Bregman iteration algorithm has advantages over the TV algorithm mainly in the following aspects: firstly, split Bregman algorithm decomposes the energy functional into two subproblems of alternating iterations variables, which avoid to solve the solution of complex curvature terms in the gradient descent equation; Secondly, split Bregman iteration algorithm not only converges faster than TV, but also has higher computational efficiency. Finally, the split Bregman iteration algorithm reduces the influence of the punishing parameter and the energy gradient to the convergence state in the image processing by using fewer iteration times than the traditional algorithm. The proposed model and algorithm can be extended to further application in image processing.

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