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A Lower-side Attainment Degree Approach for Bilevel Optimization under Uncertainty

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Abstract. *In this study, a new solution approach based on the lower-side attainment degree is developed for bilevel linear programming problems with fuzzy coefficients in both objective functions and constraint functions. In order to handle fuzzy uncertainties, we adopt the lower-side attainment degree to defuzzify fuzzy terms, and convert the fuzzy bilevel programming problem into the equivalent deterministic bilevel one. Compared with some traditional defuzzifying techniques, this kind of transformation does not produce complicated intermediate models and complex computation process, and provides a simple deterministic bilevel linear model. The resulting bilevel linear model is coped with by the extended Kth-best approach. Furthermore, we extend the developed approach to deal with the fuzzy random bilevel programming problem with the aid of expectation. Finally, we provide several numerical examples to demonstrate the feasibility and efficiency of the proposed method.*

Keywords: Bilevel optimization; Fuzzy number; Fuzzy bilevel programming; Fuzzy random variable; Lower-side attainment degree

1. **Introduction.** Bilevel optimization is a very hot research topic in mathematical programming that has attracted extensive attention from many researchers in the past few decades. So far, a large number of works about theories, algorithms and applications for bilevel optimization have been done, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The conventional bilevel programming problem is well-defined, that is to say, all the parameters involved in the problem are supposed to be known precisely. Nevertheless, some relevant parameters in many real-life bilevel problems often meet with uncertainties. Therefore, there is a growing demand to introduce any appropriate uncertainty programming technique in bilevel optimization to cope with such bilevel decision problems with uncertainty.

Fuzzy set theory proposed by Zadeh [11] is one of the most powerful tools for tackling uncertainty in optimization problems. On the basis of fuzzy set theory, uncertain parameters in bilevel optimization are characterized as fuzzy numbers, and thus a fuzzy bilevel programming problem occurs. Compared to the conventional bilevel optimization, solving such a type of problem is much trickier. Zhang et al. [12] employed $\lambda-$level sets of fuzzy numbers to change the fuzzy bilevel linear optimization problem into a multiobjective bilevel programming problem, and then developed the fuzzy Kuhn-Tucker approach, the fuzzy Kth-best approach and the fuzzy branch-and-bound approach to solve the resulting model. Hamidi and Nehi [13] introduced $\lambda-$cut to convert a bilevel linear programming with fuzzy parameters into an interval bilevel linear programming problem, and developed a solution algorithm to find a linear piecewise trapezoidal approximate fuzzy number for the upper level objective function of the fuzzy bilevel programming problem. It should be noted here that these two studies perform the defuzzifying process by discretizing fuzzy sets via $\lambda-$level sets. Katagiri et al. [14] introduced possibilistic Stackelberg problem for the fuzzy bilevel programming problem based on possibility theory, and employed linear or nonlinear bilevel programming techniques to solve the resulting deterministic bilevel model. More recently, Ren and Wang [15] proposed a new approach by combining the nearest interval approximation with reliability-based possibility degree of interval for dealing with the fuzzy bilevel linear programming problem. Besides, Zhang et al. [16] provided a survey which covered theoretical developments and applications about fuzzy bilevel decision-making techniques.

Furthermore, the practical bilevel decision-making problem may appear in a hybrid uncertain environment which contains not only a fuzzy circumstance but also a random situation. In such a case, consideration of both fuzziness and randomness in some parameters is desirable, and hence a fuzzy random bilevel programming problem arises by regarding uncertain parameters as fuzzy random variables introduced by Kwakernaak [17]. In the latest years, much attention has been devoted to efficient solution methodologies for this kind of problem. Sakawa and Katagiri [18] employed level sets and fractile criterion optimization for coping with the fuzzy random bilevel linear programming problem. Utilizing level sets and probability maximization, Sakawa and Matsui [19] suggested an interactive fuzzy programming technique to derive a satisfactory solution for the same fuzzy random bilevel programming problem under a cooperative situation. After that, Ren and Wang [20] used an interval programming approach based on level sets to reduce the fuzzy random bilevel programming problem into an equivalent crisp multiobjective bilevel one, and developed a computational methodology for finding optimistic Stackelberg solutions. Singh and Chakraborty [21] adopted the concept of fuzzy expectation and fuzzy variance to convert the fuzzy random bilevel programming problem into a fuzzy programming at first stage, and applied the aspiration level and $\alpha-$cut of the leader's objective function to transform the fuzzy programming problem into the deterministic problem. Notice that these above works are concentrated on employing $\alpha-$level sets of fuzzy numbers to carry out the defuzzifying process, and thus additional constraints and variables may be created. In addition, other latest researches on the fuzzy random bilevel programming problem may refer to [22, 23, 24, 25].

Different from $\alpha-$level set method or other traditional defuzzifying techniques, Hop [26] proposed the lower-side attainment degree to address fuzzy uncertainties in the optimization problems. In essential, this technique makes use of the relative relationship among fuzzy numbers or fuzzy random variables to execute the defuzzifying process. From the computational point of view, this method has high computational efficiency.

The main aim of this paper is to propose a new method based on the lower-side attainment degree for solving the fuzzy (random) bilevel linear programming problem. In order to do so, the lower-side attainment degree is first used to tackle fuzzy uncertainties in both objective functions and constraints, and then the fuzzy bilevel programming problem is converted into the corresponding deterministic bilevel one. Next, a new concept of Stackelberg solution on the basis of the lower-side attainment degree is introduced. Considering that the resulting deterministic model is a simple bilevel linear programming model, the extended Kth-best approach [27] is employed to deal with it. Furthermore, we extend the proposed approach to tackle the fuzzy random bilevel linear programming problem with the help of the expected value. The proposed approach in this paper produces no complex intermediate models and obtains a simple resulting deterministic model. Finally, three numerical examples are given to show the feasibility and efficiency of the developed method.

This paper is organized as follows: Section 2 recalls the concepts of triangular fuzzy number, fuzzy random variable and the lower-side attainment degree. Section 3 develops a solution methodology based on the lower-side attainment degree to deal with the fuzzy (random) bilevel linear programming problem. Section 4 gives experimental results of several numerical examples. Finally, concluding remarks are made in Section 5.

2. **Preliminaries.** In this section, the basic concepts of triangular fuzzy number and fuzzy random variable are recalled, and then the definitions and useful results relevant to the lower-side attainment degree of two fuzzy numbers or two fuzzy random variables are introduced.

Definition 2.1. ([28]) *A triangular fuzzy number $\tilde{x} = (x, l, r)$, $l, r \geq 0$, is defined as follows*

$$\mu_{\tilde{x}}(t) = \begin{cases} \max\{0, 1 - \frac{x-t}{l}\}, & t \leq x, \\ 1, & l = 0, r = 0, x = t, \\ \max\{0, 1 - \frac{t-x}{r}\}, & t \geq x, \\ 0, & \text{otherwise}, \end{cases}$$

where $l, r \geq 0 (l, r \in R)$ are the left and right spreads, respectively. In particular, a crisp number $x \in R$ can be denoted as a triangular fuzzy number $\tilde{x} = (x, 0, 0)$.

Definition 2.2. ([29]) *Suppose that $(\Omega, \mathcal{A}, P)$ be a probability space in which Ω is the sample space, $\mathcal{A}$ is Borel $\sigma-$algebra on Ω and P is the probability measure. Let $F_0(R)$ denote the set of all fuzzy numbers with compact supports on R. A fuzzy random variable is a map:*

$$\tilde{\bar{X}} : \Omega \rightarrow F_0(R), \omega \rightarrow \tilde{X}_\omega,$$

such that for any Borel set B of R and for every $\alpha \in (0, 1)$

$$\tilde{\bar{X}}_\alpha^{-1}(B) = \{\omega \in \Omega | \tilde{X}_\omega^\alpha \subset B\} \in \mathcal{A},$$

where $\tilde{X}_\omega^\alpha$ is the $\alpha-$level set of the fuzzy set $\tilde{X}_\omega$.

According to Hop [26], for two fuzzy numbers $\tilde{u}$, $\tilde{v}$ and $\tilde{u} \leq \tilde{v}$, when the intersection between the right side of $\tilde{u}$ and the left side of $\tilde{v}$ exists, the lower-side attainment degree of $\tilde{u}$ to $\tilde{v}$ can be defined as

$$D(\tilde{u}, \tilde{v}) = \int_0^1 \max\{0, \sup\{s \in R : \tilde{u}(s) \geq \alpha\} - \inf\{r \in R : \tilde{v}(r) \geq \alpha\}\} d\alpha.$$

Proposition 2.1. ([26]) *For two triangular fuzzy numbers $\tilde{u} = (u, a, b)$, $\tilde{v} = (v, c, d)$ and $u \leq v$, the average lower-side attainment degree of $\tilde{u}$ to $\tilde{v}$ is*

$$\bar{D}(\tilde{u}, \tilde{v}) = \frac{u - v + b + c}{2}.$$

Similarly, for two fuzzy random variables $\tilde{\bar{u}}$, $\tilde{\bar{v}}$ and $\tilde{\bar{u}} \leq \tilde{\bar{v}}$, Hop [26] gave the concept of the lower-side attainment degree of $\tilde{\bar{u}}$ to $\tilde{\bar{v}}$ as follows:

$$D(\tilde{\bar{u}}, \tilde{\bar{v}}) = \int_0^1 \max\{0, \sup\{s \in R : \tilde{u}_\omega(s) \geq \alpha\} - \inf\{r \in R : \tilde{v}_\omega(r) \geq \alpha\}\} d\alpha.$$

Proposition 2.2. ([26]) *Let $\tilde{\bar{u}}$ and $\tilde{\bar{v}}$ be two triangular fuzzy random variables, and $\tilde{\bar{u}} \leq \tilde{\bar{v}}$. The average lower-side attainment degree of $\tilde{\bar{u}}$ to $\tilde{\bar{v}}$ is*

$$\bar{D}(\tilde{\bar{u}}, \tilde{\bar{v}}) = \frac{u(\omega) - v(\omega) + b(\omega) + c(\omega)}{2}.$$

3. **Methodology.** In this section, a novel solution strategy based on the average lower-side attainment degree is developed to deal with the fuzzy (random) bilevel linear programming problem.

3.1. **Bilevel linear optimization with fuzzy parameters.** A general fuzzy bilevel linear programming model can be formulated as follows:

$$\begin{cases} \min\limits_{x_1} & \tilde{F}(x_1, x_2) = \tilde{c}_{11}x_1 + \tilde{c}_{12}x_2 \\ \text{s.t.} & \tilde{a}_{1i1}x_1 + \tilde{a}_{1i2}x_2 \leq \tilde{b}_{1i}, i = 1, 2, \cdots, m_1, \\ & \text{where } x_2 \text{ solves} \\ \min\limits_{x_2} & \tilde{f}(x_1, x_2) = \tilde{c}_{21}x_1 + \tilde{c}_{22}x_2 \\ \text{s.t.} & \tilde{a}_{2i1}x_1 + \tilde{a}_{2i2}x_2 \leq \tilde{b}_{2i}, i = 1, 2, \cdots, m_2, \\ & x_1 \geq 0, x_2 \geq 0, \end{cases} \tag{1}$$

where $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are vectors of decision variables controlled by the upper and lower level decision makers, respectively. $\tilde{F}(x_1, x_2)$ and $\tilde{f}(x_1, x_2)$ are the upper and lower level objective functions, respectively. $\tilde{c}_{lj}$ and $\tilde{a}_{lij}$, $l, j = 1, 2$, $i = 1, 2, \cdots, m_l$, are n_j-dimensional fuzzy vectors whose components are fuzzy numbers, and $\tilde{b}_{li}$ are fuzzy numbers.

Taking into account that the triangular fuzzy number is the most popular and commonly used type of fuzzy number in practice due to its simplicity, and we assume that fuzzy coefficients involved in problem (1) are triangular forms in this study.

Owing to fuzziness inherent in model (1), some conventional bilevel programming techniques are unable to be directly utilized for dealing with such a problem. One of the most critical issues for handling this type of problem is to remove fuzziness with the aim of converting the fuzzy bilevel programming problem into its crisp equivalent form. To this end, a solution approach on the basis of the average lower-side attainment degree is suggested to tackle problem (1) in this paper.

The lower level programming problem of problem (1) is fundamentally one single level fuzzy optimization problem. For this kind of problem, some methods based on ranking fuzzy numbers or $\alpha-$level set [30, 31] have been put forward to eliminate fuzziness contained in the problem. From a standpoint of computation, these techniques may suffer from complicated computation in the defuzzifying process. As a different approach, the lower-side attainment degree of fuzzy numbers introduced by Hop [26] can efficiently tackle fuzzy terms into the corresponding deterministic ones [26] with simple computation process. In the light of this fact, we first deal with fuzziness in the lower level programming problem by employing the lower-side attainment degree.

For a given x_1, an equivalent form of the lower level programming problem of problem (1) can be formulated by converting the lower level objective function into its equivalent fuzzy constraint:

$$\begin{cases} \min\limits_{x_2} & \theta_2 \\ \text{s.t.} & \tilde{f}(x_1, x_2) \leq \theta_2, \\ & \tilde{a}_{2i1}x_1 + \tilde{a}_{2i2}x_2 \leq \tilde{b}_{2i}, i = 1, 2, \cdots, m_2, \\ & x_2 \geq 0, \end{cases} \tag{2}$$

where θ_2 is a crisp number, fuzzified as $(\theta_2, 0, 0)$.

For convenience, denote $\tilde{g}_{2i}(x_1, x_2) = \tilde{a}_{2i1}x_1 + \tilde{a}_{2i2}x_2, i = 1, 2, \cdots, m_2$. Using arithmetic operations between triangular fuzzy numbers, the objective function value at the lower level $\tilde{f}(x_1, x_2)$ and the left hand side $\tilde{g}_{2i}(x_1, x_2)$ of the $i-$th constraint in problem (2) are triangular fuzzy numbers. Denote $\tilde{f}(x_1, x_2) = (f(x_1, x_2), l^f, r^f)$ and $\tilde{g}_{2i}(x_1, x_2) = (g_{2i}(x_1, x_2), l^{g_{2i}}, r^{g_{2i}})$.

For problem (2), we use the average lower-side attainment degree to transform fuzzy constraints into the corresponding crisp ones. Besides, the inequality conditions on the left and right hand sides of all fuzzy constraints need to be satisfied by their most possible values. Based on these discussions, problem (2) can be defuzzified into the following crisp problem by minimizing the achievement of the left-hand side to right-hand side of each constraint:

$$\begin{cases} \min\limits_{x_2} & \theta_2 + \lambda_2 + \sum\limits_{i=1}^{m_2} \eta_{2i} \\ \text{s.t.} & f(x_1, x_2) \le \theta_2, \\ & g_{2i}(x_1, x_2) \le b_{2i}, i = 1, 2, \cdots, m_2, \\ & \bar{D}(\tilde{f}(x_1, x_2), \theta_2) = \lambda_2, \\ & \bar{D}(\tilde{g}_{2i}(x_1, x_2), \tilde{b}_{2i}) = \eta_{2i}, i = 1, 2, \cdots, m_2, \\ & \lambda_2 \ge 0, \eta_{2i} \ge 0, i = 1, 2, \cdots, m_2, \\ & x_2 \ge 0. \end{cases} \tag{3}$$

For any given x_1, let $M_{\bar{D}}(x_1)$ be the set of optimal solutions of problem (3).

Next, we introduce a crisp variable θ_1 and equivalently convert the upper level objective function into the corresponding fuzzy constraint. Thus problem (1) can be rewritten as

$$\begin{cases} \min\limits_{x_1} & \theta_1 \\ \text{s.t.} & \tilde{F}(x_1, x_2) \le \theta_1, \\ & \tilde{a}_{1i1}x_1 + \tilde{a}_{1i2}x_2 \le \tilde{b}_{1i}, i = 1, 2, \cdots, m_1, \\ & x_1 \ge 0, x_2 \in M_{\bar{D}}(x_1). \end{cases} \tag{4}$$

Denote $\tilde{g}_{1i}(x_1, x_2) = \tilde{a}_{1i1}x_1 + \tilde{a}_{1i2}x_2, i = 1, 2, \cdots, m_1$. Obviously, $\tilde{F}(x_1, x_2)$ and $\tilde{g}_{1i}(x_1, x_2)$ are also two triangular fuzzy numbers. Denote $\tilde{F}(x_1, x_2) = (F(x_1, x_2), l^F, r^F)$ and $\tilde{g}_{1i} = (g_{1i}(x_1, x_2), l^{g_{1i}}, r^{g_{1i}})$.

Then the corresponding deterministic model for problem (4) can be obtained through the average lower-side attainment degree:

$$\begin{cases} \min\limits_{x_1} & \theta_1 + \lambda_1 + \sum\limits_{i=1}^{m_1} \eta_{1i} + \sum\limits_{i=1}^{m_2} \eta_{2i} \\ \text{s.t.} & F(x_1, x_2) \le \theta_1, \\ & g_{1i}(x_1, x_2) \le b_{1i}, i = 1, 2, \cdots, m_1, \\ & \bar{D}(\tilde{F}(x_1, x_2), \theta_1) = \lambda_1, \\ & \bar{D}(\tilde{g}_{1i}(x_1, x_2), \tilde{b}_{1i}) = \eta_{1i}, i = 1, 2, \cdots, m_1, \\ & x_1 \ge 0, x_2 \in M_{\bar{D}}(x_1), \lambda_1 \ge 0, \eta_{1i} \ge 0, i = 1, 2, \cdots, m_1. \end{cases} \tag{5}$$

Equivalently, the above problem can be rewritten as:

$$\left\{\begin{array}{ll}\min\limits_{x_1} & \theta_1+\lambda_1+\sum\limits_{i=1}^{m_1}\eta_{1i}+\sum\limits_{i=1}^{m_2}\eta_{2i}\\ \text{s.t.} & F(x_1,x_2)\le\theta_1,\\ & g_{1i}(x_1,x_2)\le b_{1i}, i=1,2,\cdots,m_1,\\ & \bar{D}(\tilde{F}(x_1,x_2),\theta_1)=\lambda_1,\\ & \bar{D}(\tilde{g}_{1i}(x_1,x_2),\tilde{b}_{1i})=\eta_{1i}, i=1,2,\cdots,m_1,\\ & \text{where } x_2 \text{ solves}\\ \min\limits_{x_2} & \theta_2+\lambda_2+\sum\limits_{i=1}^{m_2}\eta_{2i}\\ \text{s.t.} & f(x_1,x_2)\le\theta_2,\\ & g_{2i}(x_1,x_2)\le b_{2i}, i=1,2,\cdots,m_2,\\ & \bar{D}(\tilde{f}(x_1,x_2),\theta_2)=\lambda_2,\\ & \bar{D}(\tilde{g}_{2i}(x_1,x_2),\tilde{b}_{2i})=\eta_{2i}, i=1,2,\cdots,m_2,\\ & \lambda_1\ge 0,\eta_{1i}\ge 0, i=1,2,\cdots,m_1,\lambda_2\ge 0,\eta_{2i}\ge 0, i=1,2,\cdots,m_2,\\ & x_1\ge 0, x_2\ge 0.\end{array}\right. \tag{6}$$

It should be noticed here that the average lower-side attainment degree helps us transform the fuzzy bilevel programming problem into a crisp bilevel linear programming one. Obviously, this method provides a simple conversion process and a simple deterministic model.

Next, we give the concept of the optimal solution for the fuzzy bilevel programming problem (1).

Let $S_{\bar{D}}$ be the feasible region of problem (6).

Definition 3.1. *A point $(x_1^*, x_2^*)\in S_{\bar{D}}$ is called a lower-side attainment degree Stackelberg solution to the fuzzy bilevel programming problem (1), if (x_1^*, x_2^*) is a Stackelberg solution to problem (6).*

It is obvious that linear constraint functions are involved in the upper level programming problem, the extended Kth-best approach [27] is employed to solve the bilevel linear programming problem (6) by searching extreme points on the constraint region.

3.2. **Bilevel linear optimization under fuzzy random uncertainty.** In this section, we will extend the proposed approach based on the average lower-side attainment degree to deal with a kind of fuzzy random bilevel linear programming problem.

A fuzzy random bilevel linear programming problem in which fuzzy random variable coefficients exist in both objective functions as well as the constraints can be stated as follows:

$$\left\{\begin{array}{ll}\min\limits_{x_1} & \bar{\tilde{F}}(x_1,x_2)=\bar{\tilde{c}}_{11}x_1+\bar{\tilde{c}}_{12}x_2\\ & \text{where } x_2 \text{ solves}\\ \min\limits_{x_2} & \bar{\tilde{f}}(x_1,x_2)=\bar{\tilde{c}}_{21}x_1+\bar{\tilde{c}}_{22}x_2\\ \text{s.t.} & \bar{\tilde{a}}_{i1}x_1+\bar{\tilde{a}}_{i2}x_2\le\bar{\tilde{b}}_i, i=1,2,\cdots,s,\\ & a_{r1}x_1+a_{r2}x_2\le b_r, r=1,2,\cdots,t,\\ & x_1\ge 0, x_2\ge 0,\end{array}\right. \tag{7}$$

where $\bar{\tilde{c}}_{lj}$ and $\bar{\tilde{a}}_{ij}$, $l,j=1,2$, $i=1,2,\cdots,s$, are n_j-dimensional fuzzy random vectors whose elements are fuzzy random variables, and $\bar{\tilde{b}}_i$ are fuzzy random variables; a_{rj}, $r=1,2,\cdots,t$, are n_j-dimensional crisp vectors, and b_r are constants.

Assume that the triangular form of fuzzy random variable is considered for the uncertain parameters in problem (7). Denote $\tilde{\bar{g}}_i(x_1, x_2) = \tilde{\bar{a}}_{i1}x_1 + \tilde{\bar{a}}_{i2}x_2, i = 1, 2, \cdots, s$. According to Zadeh's extension principle [32], the upper and lower level objective functions and the left hand sides of all uncertain constraints become triangular fuzzy random variables. Denote $\tilde{\bar{F}}_\omega(x_1, x_2) = (F_\omega(x_1, x_2), l^{F_\omega}, r^{F_\omega})$, $\tilde{\bar{f}}_\omega(x_1, x_2) = (f_\omega(x_1, x_2), l^{f_\omega}, r^{f_\omega})$, $(\tilde{\bar{g}}_i)_\omega(x_1, x_2) = ((g_i)_\omega(x_1, x_2), l^{(g_i)_\omega}, r^{(g_i)_\omega})$, and $(\tilde{\bar{b}}_i)_\omega = ((b_i)_\omega, l^{(b_i)_\omega}, r^{(b_i)_\omega})$, $\forall \omega \in \Omega$.

Currently, the frequently used approach to tackle the fuzzy random bilevel programming problem is to defuzzify and derandomize fuzzy random terms for the sake of reducing the problem into the deterministic one dealt with by some efficient solution strategies. Here the defuzzifying process utilizes the average lower-side attainment degree technique and the derandomizing process adopts expectation method.

We first introduce crisp variables θ_1, θ_2 and convert the upper and lower level objective functions into their equivalent fuzzy random constraints, and then obtain its equivalent form:

$$\begin{cases} \min\limits_{x_1} & \theta_1 \\ \text{s.t.} & \tilde{\bar{F}}(x_1, x_2) \leq \theta_1, \\ & \text{where } x_2 \text{ solves} \\ \min\limits_{x_2} & \theta_2 \\ \text{s.t.} & \tilde{\bar{f}}(x_1, x_2) \leq \theta_2, \\ & \tilde{\bar{a}}_{i1}x_1 + \tilde{\bar{a}}_{i2}x_2 \leq \tilde{\bar{b}}_i, i = 1, 2, \cdots, s, \\ & a_{r1}x_1 + a_{r2}x_2 \leq b_r, r = 1, 2, \cdots, t, \\ & x_1 \geq 0, x_2 \geq 0. \end{cases} \tag{8}$$

Then we apply the average lower-side attainment degree and the expected value to convert fuzzy random constraints into deterministic constraints, and thus reduce model (8) into a deterministic bilevel programming problem as follows:

$$\begin{cases} \min\limits_{x_1} & \theta_1 + E[\lambda_1(\omega)] + E(\sum\limits_{i=1}^{s}[\eta_i(\omega)]) \\ \text{s.t.} & F_\omega(x_1, x_2) \leq \theta_1, \\ & \bar{D}(\tilde{\bar{F}}_\omega(x_1, x_2), \theta_1) = \lambda_1(\omega), \\ & \text{where } x_2 \text{ solves} \\ \min\limits_{x_2} & \theta_2 + E[\lambda_2(\omega)] + E(\sum\limits_{i=1}^{s}[\eta_i(\omega)]) \\ \text{s.t.} & f_\omega(x_1, x_2) \leq \theta_2, \\ & (g_i)_\omega(x_1, x_2) \leq (b_i)_\omega, i = 1, 2, \cdots, s, \\ & \bar{D}(\tilde{\bar{f}}_\omega(x_1, x_2), \theta_2) = \lambda_2(\omega), \\ & \bar{D}((\tilde{\bar{g}}_i)_\omega(x_1, x_2), (\tilde{\bar{b}}_i)_\omega) = \eta_i(\omega), i = 1, 2, \cdots, s, \\ & a_{r1}x_1 + a_{r2}x_2 \leq b_r, r = 1, 2, \cdots, t, \\ & \lambda_1(\omega) \geq 0, \lambda_2(\omega) \geq 0, \eta_i(\omega) \geq 0, \omega \in \Omega, i = 1, 2, \cdots, s, \\ & x_1 \geq 0, x_2 \geq 0, \end{cases} \tag{9}$$

where E represents the expected value.

Let $S_{\bar{D}E}$ be the feasible region of problem (9).

Definition 3.2. *If $(x_1^*, x_2^*) \in S_{\bar{D}E}$ is a Stackelberg solution to problem (9), then (x_1^*, x_2^*) is called an expected lower-side attainment degree Stackelberg solution to the fuzzy random bilevel programming problem (7).*

It is evident that the transformed model (9) is also a bilevel linear programming problem with linear constraint functions in the upper level programming. Similar to the previous section,we use the extended Kth-best approach to solve model (9).

4. **Numerical examples.** In this section, three numerical examples are provided to show the effectiveness of the proposed technique for the fuzzy (random) bilevel linear programming problem. Furthermore, some comparisons and discussions are given to further illustrate the developed approach.

Example 4.1. *Consider a fuzzy bilevel linear programming problem taken from Hamidi and Nehi* [13]*:*

$$\begin{cases} \min\limits_{x\geq 0} & \tilde{F}(x,y) = \tilde{1}x - \tilde{4}y \\ & \text{where } y \text{ solves} \\ \min\limits_{y\geq 0} & \tilde{f}(x,y) = \tilde{1}y \\ \text{s.t.} & \tilde{2}x - \tilde{1}y \geq \tilde{0}, \\ & -\tilde{2}x - \tilde{1}y \geq -\tilde{12}, \\ & \tilde{3}x - \tilde{2}y \geq \tilde{4}, \end{cases} \tag{10}$$

where all fuzzy coefficients are assumed to be in the form of triangular fuzzy number $\tilde{t} = (t, 1, 1)$.

According to model (6), problem (10) can be transformed into the following bilevel model:

$$\begin{cases} \min & \theta_1 + \lambda_1 + \eta_1 + \eta_2 + \eta_3 \\ \text{s.t.} & x - 4y \leq \theta_1, \\ & \bar{D}(\tilde{1}x - \tilde{4}y, \theta_1) = \lambda_1, \\ & \text{where } y \text{ solves} \\ \min & \theta_2 + \lambda_2 + \eta_1 + \eta_2 + \eta_3 \\ \text{s.t.} & y \leq \theta_2, \\ & 2x - y \geq 0, \\ & -2x - y \geq -12, \\ & 3x - 2y \geq 4, \\ & \bar{D}(\tilde{1}y, \theta_2) = \lambda_2, \\ & \bar{D}(\tilde{0}, \tilde{2}x - \tilde{1}y) = \eta_1, \\ & \bar{D}(-\tilde{12}, -\tilde{2}x - \tilde{1}y) = \eta_2, \\ & \bar{D}(\tilde{4}, \tilde{3}x - \tilde{2}y) = \eta_3, \\ & x \geq 0, y \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \eta_1 \geq 0, \eta_2 \geq 0, \eta_3 \geq 0. \end{cases} \tag{11}$$

Then we obtain the following bilevel linear programming problem by calculating all average lower-side attainment degrees:

$$\left\{\begin{array}{ll}\min & \theta_1+\lambda_1+\eta_1+\eta_2+\eta_3\\ \text{s.t.} & x-4y\le\theta_1,\\ & \frac{1}{2}(x-4y-\theta_1+x+y+0)=\lambda_1,\\ & \text{where } y \text{ solves}\\ \min & \theta_2+\lambda_2+\eta_1+\eta_2+\eta_3\\ \text{s.t.} & y\le\theta_2,\\ & 2x-y\ge 0,\\ & -2x-y\ge -12,\\ & 3x-2y\ge 4,\\ & \frac{1}{2}(y-\theta_2+y+0)=\lambda_2,\\ & \frac{1}{2}(0-(2x-y)+1+x+y)=\eta_1,\\ & \frac{1}{2}(-12-(-2x-y)+1+x+y)=\eta_2,\\ & \frac{1}{2}(4-(3x-2y)+1+x+y)=\eta_3,\\ & x\ge 0, y\ge 0, \lambda_1\ge 0, \lambda_2\ge 0, \eta_1\ge 0, \eta_2\ge 0, \eta_3\ge 0.\end{array}\right. \tag{12}$$

Now solving problem (12) by the extended Kth-best approach, we can obtain the optimal solution $(x^*, y^*) = (3.0, 1.0)$. Put this solution into the upper level objection function of problem (10), the corresponding objection function value is $\tilde{F}^* = (-1, 4, 4)$.

For this example, Hamidi and Nehi [13] applied $\lambda-$cut as the defuzzifying technique to construct an interval bilevel programming model from a fuzzy one. As far as the resulting model is concerned, our method provides a simple crisp bilevel linear programming model, which can be coped with easily by some classical solution strategies.

Example 4.2. *Consider the following example taken from Zhang and Lu* [33]*:*

$$\left\{\begin{array}{ll}\min\limits_{x\ge 0} & \tilde{1}x-\tilde{2}y\\ \text{s.t.} & -\tilde{1}x+\tilde{3}y\le\tilde{4},\\ & \text{where } y \text{ solves}\\ \min\limits_{y\ge 0} & \tilde{1}x+\tilde{1}y\\ \text{s.t.} & \tilde{1}x-\tilde{1}y\le\tilde{0},\\ & -\tilde{1}x-\tilde{1}y\le\tilde{0},\end{array}\right. \tag{13}$$

where all the coefficients are triangular fuzzy numbers, denoted by $\tilde{t} = (t, 1, 1)$.

Using model (6), problem (13) can be transferred to a bilevel linear programming problem as follow:

$$\left\{\begin{array}{ll}\min & \theta_1+\lambda_1+\eta_{11}+\eta_{21}+\eta_{22}\\ \text{s.t.} & x-2y\le\theta_1,\\ & -x+3y\le 4,\\ & \frac{1}{2}(x-2y-\theta_1+x+y+0)=\lambda_1,\\ & \frac{1}{2}(-x+3y-4+x+y+1)=\eta_{11},\\ & \text{where } y \text{ solves}\\ \min & \theta_2+\lambda_2+\eta_{21}+\eta_{22}\\ \text{s.t.} & x+y\le\theta_2,\\ & x-y\le 0,\\ & -x-y\le 0,\\ & \frac{1}{2}(x+y-\theta_2+x+y+0)=\lambda_2,\\ & \frac{1}{2}(x-y-0+x+y+1)=\eta_{21},\\ & \frac{1}{2}(-x-y-0+x+y+1)=\eta_{22},\\ & x\ge 0, y\ge 0, \lambda_1\ge 0, \lambda_2\ge 0, \eta_{11}\ge 0, \eta_{21}\ge 0, \eta_{22}\ge 0.\end{array}\right. \tag{14}$$

With the help of the extended Kth-best method, the obtained optimal solution is $(x^*, y^*) = (0.3750, 0.7500)$, and the corresponding objective function value at the upper level of problem (13) is $\tilde{F}^* = (-1.125, 1.125, 1.125)$.

To solve this example, Zhang and Lu [33] employed $\lambda-$cut to change problem (13) into a multiobjective bilevel programming model. From the resulting deterministic model, it is apparent that a bilevel linear programming model with a single objective function at each level constructed by our approach is usually easier to tackle than a multiobjective bilevel programming model constructed by Zhang and Lu' method. In addition, the optimal solution obtained by our approach is extremely different from that obtained by Zhang and Lu' method. It is mainly because different concepts of optimal solution are defined based on different perspectives of decision-making under an uncertain environment by our approach and Zhang and Lu' method, leading to different optimal results.

Example 4.3. *We consider the following fuzzy random bilevel linear programming problem in* [34]*:*

$$\begin{cases} \min\limits_{x_1} & \tilde{\bar{z}}_1(x_1,x_2) = \tilde{\bar{c}}_{111}x_{11} + \tilde{\bar{c}}_{112}x_{12} + \tilde{\bar{c}}_{113}x_{13} + \tilde{\bar{c}}_{121}x_{21} + \tilde{\bar{c}}_{122}x_{22} + +\tilde{\bar{c}}_{123}x_{23} \\ & \text{where } x_2 \text{ solves} \\ \min\limits_{x_2} & \tilde{\bar{z}}_2(x_1,x_2) = \tilde{\bar{c}}_{211}x_{11} + \tilde{\bar{c}}_{212}x_{12} + \tilde{\bar{c}}_{213}x_{13} + \tilde{\bar{c}}_{221}x_{21} + \tilde{\bar{c}}_{222}x_{22} + +\tilde{\bar{c}}_{223}x_{23} \\ \text{s.t.} & 2x_{11} + 3x_{12} + x_{13} + 2x_{21} + 3x_{22} + 3x_{23} \le 65, \\ & 4x_{11} + 4x_{12} + 2x_{13} + 3x_{21} + 2x_{22} + x_{23} \le 80, \\ & 2x_{11} + 4x_{12} + 3x_{13} + 3x_{21} + 2x_{22} + 2x_{23} \le 105, \\ & -3x_{11} - 2x_{12} - 2x_{13} - 4x_{21} - x_{22} - 4x_{23} \le -70, \\ & x_1 = (x_{11}, x_{12}, x_{13})^{\mathrm{T}} \ge 0, x_2 = (x_{21}, x_{22}, x_{23})^{\mathrm{T}} \ge 0, \end{cases} \tag{15}$$

where all the coefficients in both objective functions are triangular fuzzy random variables. Tables 1 and 2 give the values of these coefficients.

Table 1. Values of coefficients in the upper level objective function

Probability	$\tilde{\bar{c}}_{111}$	$\tilde{\bar{c}}_{112}$	$\tilde{\bar{c}}_{113}$	$\tilde{\bar{c}}_{121}$	$\tilde{\bar{c}}_{122}$	$\tilde{\bar{c}}_{123}$
$P(\omega_1) =$ 0.25	(2.3,0.8,0.8)	(-1.0,1.2,1.1)	(1.3,0.7,0.5)	(-1.3,0.9,0.6)	(-1.8,1.3,0.9)	(2.0,0.6,1.0)
$P(\omega_2) =$ 0.40	(2.0,0.8,0.8)	(-1.3,1.2,1.1)	(2.0,0.7,0.5)	(1.1,0.9,0.6)	(-2.1,1.3,0.9)	(2.4,0.6,1.0)
$P(\omega_3) =$ 0.35	(1.9,0.8,0.8)	(-2.4,1.2,1.1)	(2.7,0.7,0.5)	(-1.5,0.9,0.6)	(-1.2,1.3,0.9)	(3.8,0.6,1.0)

Table 2. Values of coefficients in the lower level objective function

Probability	$\tilde{\bar{c}}_{211}$	$\tilde{\bar{c}}_{212}$	$\tilde{\bar{c}}_{213}$	$\tilde{\bar{c}}_{221}$	$\tilde{\bar{c}}_{222}$	$\tilde{\bar{c}}_{223}$
$P(\omega_1) =$ 0.45	(3.0,0.7,0.7)	(1.7,1.2,0.9)	(-1.6,0.8,0.6)	(-1.4,0.5,1.0)	(-1.6,0.9,0.8)	(1.7,1.1,0.9)
$P(\omega_2) =$ 0.15	(1.7,0.7,0.7)	(1.3,1.2,0.9)	(-2.3,0.8,0.6)	(-0.8,0.5,1.0)	(-1.9,0.9,0.8)	(2.6,1.1,0.9)
$P(\omega_3) =$ 0.40	(2.3,0.7,0.7)	(0.9,1.2,0.9)	(-1.0,0.8,0.6)	(-2.0,0.5,1.0)	(-1.2,0.9,0.8)	(3.5,1.1,0.9)

For simplicity, let S denote the constraint region of problem (15).

Now we first transform the upper and lower level objective functions into their equivalent fuzzy random constraints by introducing two crisp variables θ_1 and θ_2, and then the equivalent model of problem (15) is:

$$\begin{cases} \min & \theta_1 \\ \text{s.t.} & \tilde{\bar{c}}_{111}x_{11} + \tilde{\bar{c}}_{112}x_{12} + \tilde{\bar{c}}_{113}x_{13} + \tilde{\bar{c}}_{121}x_{21} + \tilde{\bar{c}}_{122}x_{22} + \tilde{\bar{c}}_{123}x_{23} \le \theta_1, \\ & \text{where } x_2 \text{ solves} \\ \min & \theta_2 \\ \text{s.t.} & \tilde{\bar{c}}_{211}x_{11} + \tilde{\bar{c}}_{212}x_{12} + \tilde{\bar{c}}_{213}x_{13} + \tilde{\bar{c}}_{221}x_{21} + \tilde{\bar{c}}_{222}x_{22} + \tilde{\bar{c}}_{223}x_{23} \le \theta_2, \\ & (x_1, x_2) \in S. \end{cases} \tag{16}$$

According to model (9), a deterministic form of problem (16) is:

$$\begin{cases} \min & \theta_1 + 0.25\lambda_{11} + 0.40\lambda_{12} + 0.35\lambda_{13} \\ \text{s.t.} & 2.3x_{11} - x_{12} + 1.3x_{13} - 1.3x_{21} - 1.8x_{22} + 2.0x_{23} \le \theta_1, \\ & 2.0x_{11} - 1.3x_{12} + 2.0x_{13} + 1.1x_{21} - 2.1x_{22} + 2.4x_{23} \le \theta_1, \\ & 1.9x_{11} - 2.4x_{12} + 2.7x_{13} - 1.5x_{21} - 1.2x_{22} + 3.8x_{23} \le \theta_1, \\ & \frac{1}{2}(2.3x_{11} - x_{12} + 1.3x_{13} - 1.3x_{21} - 1.8x_{22} + 2.0x_{23} - \theta_1 + 0.8x_{11} + 1.1x_{12} \\ & \quad +0.5x_{13} + 0.6x_{21} + 0.9x_{22} + 1.0x_{23} + 0) = \lambda_{11}, \\ & \frac{1}{2}(2.0x_{11} - 1.3x_{12} + 2.0x_{13} + 1.1x_{21} - 2.1x_{22} + 2.4x_{23} - \theta_1 + 0.8x_{11} + 1.1x_{12} \\ & \quad +0.5x_{13} + 0.6x_{21} + 0.9x_{22} + x_{23} + 0) = \lambda_{12}, \\ & \frac{1}{2}(1.9x_{11} - 2.4x_{12} + 2.7x_{13} - 1.5x_{21} - 1.2x_{22} + 3.8x_{23} - \theta_1 + 0.8x_{11} + 1.1x_{12} \\ & \quad +0.5x_{13} + 0.6x_{21} + 0.9x_{22} + x_{23} + 0) = \lambda_{13}, \\ & \text{where } x_2 \text{ solves} \\ \min & \theta_2 + 0.45\lambda_{21} + 0.15\lambda_{22} + 0.40\lambda_{23} \\ \text{s.t.} & 3.0x_{11} + 1.7x_{12} - 1.6x_{13} - 1.4x_{21} - 1.6x_{22} + 1.7x_{23} \le \theta_2, \\ & 1.7x_{11} + 1.3x_{12} - 2.3x_{13} - 0.8x_{21} - 1.9x_{22} + 2.6x_{23} \le \theta_2, \\ & 2.3x_{11} + 0.9x_{12} - x_{13} - 2.0x_{21} - 1.2x_{22} + 3.5x_{23} \le \theta_2, \\ & \frac{1}{2}(3.0x_{11} + 1.7x_{12} - 1.6x_{13} - 1.4x_{21} - 1.6x_{22} + 1.7x_{23} - \theta_2 + 0.7x_{11} + 0.9x_{12} \\ & \quad +0.6x_{13} + x_{21} + 0.8x_{22} + 0.9x_{23} + 0) = \lambda_{21}, \\ & \frac{1}{2}(1.7x_{11} + 1.3x_{12} - 2.3x_{13} - 0.8x_{21} - 1.9x_{22} + 2.6x_{23} - \theta_2 + 0.7x_{11} + 0.9x_{12} \\ & \quad +0.6x_{13} + x_{21} + 0.8x_{22} + 0.9x_{23} + 0) = \lambda_{22}, \\ & \frac{1}{2}(2.3x_{11} + 0.9x_{12} - 1.0x_{13} - 2.0x_{21} - 1.2x_{22} + 3.5x_{23} - \theta_2 + 0.7x_{11} + 0.9x_{12} \\ & \quad +0.6x_{13} + x_{21} + 0.8x_{22} + 0.9x_{23} + 0) = \lambda_{23}, \\ & \lambda_{11} = \lambda_1(\omega_1) \ge 0, \lambda_{12} = \lambda_1(\omega_2) \ge 0, \lambda_{13} = \lambda_1(\omega_3) \ge 0, \lambda_{21} = \lambda_2(\omega_1) \ge 0, \\ & \lambda_{22} = \lambda_2(\omega_2) \ge 0, \lambda_{23} = \lambda_2(\omega_3) \ge 0, (x_1, x_2) \in S. \end{cases} \tag{17}$$

Through solving model (17), we obtain the optimal solution $(x_{11}^*, x_{12}^*, x_{13}^*, x_{21}^*, x_{22}^*, x_{23}^*) = (4.5490, 0.3788, 0, 15.4669, 6.5185, 0.8512)$.

For this example, Sakawa et al. [34] combined expectation optimization with possibility and necessity to transform and solve it. From the point of view of the resulting deterministic model, Sakawa et al.' approach gives a bilevel linear fractional programming model, and our method obtains a bilevel linear programming model which is the simplest form in bilevel programming. From the final results, both E-P-Stackelberg solution and E-N-Stackelberg solution obtained in [34] are different from the optimal solution obtained by our approach in terms of different definitions of optimal solution.

5. **Conclusion.** In this research, we study bilevel linear programming problems within a fuzzy (random) environment, where uncertain coefficients are considered as fuzzy numbers (fuzzy random variables). To effectively address fuzzy uncertainties, the lower-side attainment degree is utilized to defuzzify fuzzy terms, and then change the fuzzy bilevel programming problem into the equivalent deterministic bilevel linear problem solved by

the extended Kth-best approach. Moreover, the developed method is extended to tackle fuzzy random bilevel programming problems with the help of expectation. The advantage of the proposed approach is that it can help make simple transformation processes and obtain a simple deterministic model, and thus improve the computational efficiency.

The proposed approach can be further extended to handle multiobjective bilevel optimization problems under fuzzy (random) environments. Besides, the applicability of the proposed method is also considered as another future research work.

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REFERENCES

[1] S. Dempe, *Foundations of Bilevel Programming*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[2] B. Colson, P. Marcotte, and G. Savard, An overview of bilevel optimization, *Annals of Operations Research*, vol.153, no.1, pp.235–256, 2007.

[3] S. Dempe, and A. B. Zemkoho, The bilevel programming problem: Reformulations, constraint qualifications and optimality conditions, *Mathematical Programming*, vol.138, no.1-2, pp.447–473, 2013.

[4] S. Dempe, V. Kalashnikov, G. A. Pérez-Valdés, and N. Kalashnykova, *Bilevel Programming Problems: Theory, Algorithms and Applications to Energy Networks*, Kluwer Academic Publishers, Springer, Berlin, 2015.

[5] G. Q. Zhang, J. Lu, and Y. Gao, *Multi-level Decision Making: Models, Methods and Applications*, Springer, Berlin, 2015.

[6] G. M. Wang, L. M. Ma, and J. W. Chen, A bilevel improved fruit fly optimization algorithm for the nonlinear bilevel programming problem, *Knowledge-Based Systems*, vol.138, pp.113–123, 2017.

[7] M. M. Islam, H. K. Singh, T. Ray, and A. Sinha, An enhanced memetic algorithm for single-objective bilevel optimization problems, *Evolutionary Computation*, vol.25, no.4, pp.607–642, 2017.

[8] V. V. Kalashnikov, S. Dempe, G. A. Pérez-Valdés, N. I. Kalashnykova, and J. F. Camacho-Vallejo, Bilevel programming and applications, *Mathematical Problems in Engineering*, vol.2015, pp.16 pages, 2015.

[9] A. Sinha, P. Malo, and K. Deb, Transportation policy formulation as a multi-objective bilevel optimization problem, *2015 IEEE Congress on Evolutionary Computation (CEC-2015)*, Sendai, Japan, pp.1-8 , 2015.

[10] A. Sinha, P. Malo, and K. Deb, A review on bilevel optimization: From classical to evolutionary approaches and applications, *IEEE Transactions on Evolutionary*, vol.22, no.2, pp.276–295, 2018.

[11] L. A. Zadeh, Fuzzy sets, *Information and Control*, vol.8, no.3, pp.338–353, 1965.

[12] G. Q. Zhang, J. Lu, and T. Dillon, Fuzzy linear bilevel optimization: Solution concepts, approaches and applications, *Fuzzy Logic*, vol.215, pp.351–379, 2007.

[13] F. Hamidi, and N. Mishmast, Bilevel linear programming with fuzzy parameters, *Iranian Journal of Fuzzy Systems*, vol.10, no.4, pp.83–99, 2013.

[14] H. Katagiri, K. Kato, and T. Uno, Possibilistic Stackelberg solutions to bilevel linear programming problems with fuzzy parameters, *Journal of Intelligent & Fuzzy Systems*, vol.32, no.6, pp.4485–4501, 2017.

[15] A. H. Ren, and Y. P. Wang, An approach based on reliability-based possibility degree of interval for solving general interval bilevel linear programming problem, *Soft Computing*, pp.1–10, 2017, Online.

[16] G. Q. Zhang, J. L. Han, and J. Lu, Fuzzy bilevel decision-making techniques: A survey, *International Journal of Computational Intelligence Systems*, vol.9, pp.25–34, 2016.

[17] H. Kwakernaak, Fuzzy random variables-I: Definitions and theorems, *Information Sciences*, vol.15, no.1, pp.1–29, 1978.

[18] M. Sakawa, and H. Katagiri, Stackelberg solutions for fuzzy random two-level linear programming through level sets and fractile criterion optimization, *Central European Journal of Operations Research*, vol.20, pp.101–117, 2012.

[19] M. Sakawa, and T. Matsui, Interactive fuzzy random cooperative two-level linear programming through level sets based probability maximization, *Expert Systems with Applications*, vol.40, pp.1400–1406, 2013.

[20] A. H. Ren, Y. P. Wang, Optimistic Stackelberg solutions to bilevel linear programming with fuzzy random variable coefficients, *Knowledge-Based Systems*, vol.67, pp.206–217, 2014.

[21] V. P. Singh, and D. Chakraborty, Solving bi-level programming problem with fuzzy random variable coefficients, *Journal of Intelligent & Fuzzy Systems*, vol.32, pp.521–528, 2017.

[22] M. Sakawa, and T. Matsui, Fuzzy random non-cooperative two-level linear programming through fractile models with possibility and necessity, *Engineering Optimization*, vol.45, no.7, pp.811–833, 2013.

[23] M. Sakawa, and T. Matsui, Interactive fuzzy random two-level linear programming based on level sets and fractile criterion optimization, *Information Sciences*, vol.238, pp.163–175, 2013.

[24] A. H. Ren, Y. P. Wang, and X. X. Xue, An interval programming approach for the bilevel linear programming problem under fuzzy random environments, *Soft Computing*, vol.18, no.5, pp.995–1009, 2014.

[25] M. Sakawa, T. Matsui, Bilevel linear programming with fuzzy random variables through absolute deviation minimisation, *International Journal of Operational Research*, vol.25, no.1, pp.1–27, 2016.

[26] N. V. Hop, Solving linear programming problems under fuzziness and randomness environment using attainment values, *Information Sciences*, vol.177, no.14, pp.2971–2984, 2007.

[27] C. Shi, J. Lu, and G. Q. Zhang, An extended Kth-best approach for linear bilevel programming, *Applied Mathematics and Computation*, vol.164, no.3, pp.843–855, 2005.

[28] H. J. Zimmermann, Fuzzy set theory - and its applications, Springer Netherlands, Dordrecht, 2001. http://dx.doi.org/10.1007/978-94-010-0646-0

[29] M. K. Luhandjula, Fuzziness and randomness in an optimization framework, *Fuzzy Sets and Systems*, vol.77, pp.291–297, 1996.

[30] M. Jiménez, M. Arenas, A. Bilbao, and M. V. Rodríguez, Linear programming with fuzzy parameters: An interactive method resolution, *European Journal of Operational Research*, vol.177. no.3, pp.1599–1609, 2007.

[31] X. W. Liu, Measuring the satisfaction of constraints in fuzzy linear programming, *Fuzzy Sets and Systems*, vol.122, no.2, pp.263–275, 2001.

[32] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, vol.1, no.1, pp.3–28, 1978.

[33] G. Q. Zhang, and J. Lu, The definition of optimal solution and an extended Kuhn-Tucker approach for fuzzy linear bilevlel programming, *The IEEE Intelligent Informatics Bulletin*, vol.6, no.2, pp.1–8, 2005.

[34] M. Sakawa, H. Katagiri, and T. Matsui, Fuzzy random bilevel linear programming through expectation optimization using possibility and necessity, *International Journal of Machine Learning and Cybernetics*, vol.3, pp.183–192, 2012.