Impulsive Synchronization for Coupled Reaction-Diffusion System and its Application in Information Hiding

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***Abstract*—In this paper, basing on the existence of the global solutions for coupled reaction-diffusion system with and without impulses, impulsive synchronization of coupled reaction-diffusion system is investigated. A criterion for the solution of impulsive error system to be equi-attractive in the large is determined. A numerical example is given to illustrated the theoretical result. As an application, an information hiding scheme basing on impulsive synchronization of partial differential system is developed. A simple information signal is masked and recovered, simulation results show the method is effective.**

***Keywords*-reaction-diffusion system; impulsive synchronization; information hiding.**

I. INTRODUCTION

The theory of impulsive ordinary differential equations and its applications to the fields of science and engineering have been very active research topics[1-2]. Extending the theory of impulsive differential equations to partial differential equations has also gained considerable attention recently[3-4]. Unfortunately, there has been few theory on impulsive synchronization of PDE. Moreover, since the PDE have more complex dynamics in comparison with ODE's, using impulsive synchronization to information hiding have more advantage, there has been few work about it.

In this paper, an interaction of predator and prey species is considered, the main object of this investigation is the following two-dimensional coupled reaction-diffusion system:

$$\begin{cases} u_t - a\Delta u = f(u,v), & \text{in } \Omega\times(0,\infty) \\ v_t - c\Delta u - d\Delta v = g(u,v), & \text{in } \Omega\times(0,\infty) \\ u = v = 0, & \text{on } \partial\Omega\times(0,\infty) \\ u(x,0) = u_0(x), & \text{in } \Omega \\ v(x,0) = v_0(x), & \text{in } \Omega \end{cases} \quad (1.1)$$

where Ω is a bounded open set in R^n with smooth boundary $\partial\Omega$. This system has been introduced by Shigesada et al.[5] as a model of the population dynamics in Ω of two competitive species which move under the influence of population pressure and of environmental parameters. The functions $u(x,t)$ and $v(x,t)$determine the densities of prey and predator, respectively, at a space point x and time t. Denote by $\triangle u = \sum_{k=1}^{n} \partial^2 u/\partial x_k^2$ the Laplace operator.

Biological species can undergo discrete changes of relatively short duration at fixed timed (for examples, due to stocking or harvesting of species), Moreover, continuous changes in environmental parameters such as temperature or rainfall can also create discontinuous outbreaks in pest populations[6]. Systems with such kinds of discontinuous changes can be investigated by the theory of impulsive differential equations[7]. Therefore, there have been a number of studies which have applied the theory to biological problems[8]. The impulsive reaction-diffusion system can be described as follows:

$$\begin{cases} \left.\begin{aligned} & u_t - a\Delta u = f(u,v) \\ & v_t - c\Delta u - d\Delta v = g(u,v)) \end{aligned}\right\} t \neq t_k, x\in\Omega \\ \left.\begin{aligned} & Ju(x,t) = -q_k u, \\ & Jv(x,t) = -p_k v, \end{aligned}\right\} t = t_k, x\in\Omega \\ u(x,0) = u_0(x), v(x,0) = v_0(x), x\in\Omega \\ u = v = 0,\ x\in\partial\Omega,\ t\in R_+. \end{cases} \quad (1.2)$$

where $k = 1, 2, \ldots$, and $Ju(x,t) = u(x,t_k^+) - u(x,t_k^-)$, for all $x\in\Omega$, $u(x,t_k^+) = \lim_{t\to t_k^+} u(t,x)$, $Jv(x,t)$ has the similar definition as $Ju(x,t)$. $p_k, q_k \in R$ are constants. The moments of impulse satisfy $0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$, and $t_k \to \infty$, as $k\to\infty$. $a > 0, d > 0$, $c \in R$, $f, g : R_+^2 \to R$ are given functions which satisfy the locally Lipschitz conditions. u_0, v_0 are initial given nonnegative functions which are assumed to satisfy the following conditions:

$$(\mathbf{H}_0):\ \begin{cases} u_0, v_0 \in H^{1+\varepsilon}(\Omega), & 0<\varepsilon<1 \\ u = v = 0, & x\in\partial\Omega \end{cases}$$

For system (1.1), we have the following result about the existence of the global solutions.

Theorem 1 [9] If systems (1.1) satisfies the assumptions $(\mathbf{H}_0)$ and the following assumptions $(\mathbf{H}_1)$:

(c1) For all $u, v\in R$, $f(u,v) \le a_1 u + a_2$;

(c2) For all $u, v\in R$, $|vg(u,v)| \le C_1(u)v^2 + C_2(u)$, and $Af + Bg \le C_1(u)v + C_2(u)$;

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(c3) $f(r,s), f_r(r,s), f_s(r,s), g(r,s), g_r(r,s), g_s(r,s)$ are polynomically bounded;

(c4) $\forall u, v \geq 0, f(0,v) \geq 0, g(u,0) \geq 0$.

then systems (1.1) has an unique global solution (u,v), such that for all $0 < T < \infty$,

$$u, v \in C((0,T], H^2(\Omega)) \cap C^1((0,T], L^2(\Omega)).$$

Secondly, the existence of the global solutions for impulsive coupled reaction-diffusion system (1.2) has been obtained.

Theorem 2 [9] If the assumptions ($\mathbf{H}_0$) and ($\mathbf{H}_1$) hold, then systems (1.2) has an unique global solution (u,v), such that for all $k = 1, 2, \ldots$,

$$u, v \in C((t_k, t_{k+1}], H^2(\Omega)) \cap C^1((t_k, t_{k+1}], L^2(\Omega)).$$

where $u(t_{k+1}) = u(t_{k+1}^-), v(t_{k+1}) = v(t_{k+1}^-)$.

In this paper, on the basis of Theorem 1 and Theorem 2, the impulsive synchronization of coupled reaction-diffusion systems (1.1) is considered.

Here, system (1.1) is the driving system, and the driven system with impulsive control is given by

$$\begin{cases} \left.\begin{array}{l} \bar{u}_t - a\Delta\bar{u} = f(\bar{u},\bar{v}) \\ \bar{v}_t - c\Delta\bar{u} - d\Delta v = g(\bar{u},\bar{v}) \end{array}\right\} t \neq t_k, x \in \Omega \\ \left.\begin{array}{l} J\bar{u}(x,t) = -q_k(u-\bar{u}) \\ Jv(x,t) = -p_k(v-\bar{v}) \end{array}\right\} t = t_k, x \in \Omega \\ \bar{u}(x,0) = \bar{u}_0(x), \bar{v}(x,0) = \bar{v}_0(x), x \in \Omega \\ \bar{u} = \bar{v} = 0,\ x \in \partial\Omega,\ t \in R_+ \end{cases} \tag{1.3}$$

where f, g satisfy the following Lipschitz conditions with $L_0 > 0$:

$$\begin{aligned} & (u-\bar{u})[f(u,v) - f(\bar{u},\bar{v})] + (v-\bar{v})[g(u,v) - g(\bar{u},\bar{v})] \\ \leq\ & L_0[(u-\bar{u})^2 + (v-\bar{v})^2]. \end{aligned} \tag{1.4}$$

Let $\mathbf{e} = (e_1, e_2)^{\mathrm{T}} = (u-\bar{u}, v-\bar{v})^{\mathrm{T}}$, The following impulsive synchronization error system is obtained:

$$\begin{cases} \left.\begin{array}{l} \frac{\partial e_1}{\partial t} - a\Delta e_1 = \psi_1(e_1,e_2) \\ \frac{\partial e_2}{\partial t} - c\Delta e_1 - d\Delta e_2 = \psi_2(e_1,e_2) \end{array}\right\} t \neq t_k, x \in \Omega \\ \left.\begin{array}{l} Je_1(x,t) = q_k e_1 \\ Je_2(x,t) = p_k e_2 \end{array}\right\} t = t_k, x \in \Omega \\ e_1(x,0) = e_1^0(x), e_2(x,0) = e_2^0(x), x \in \Omega \\ e_1 = e_2 = 0,\ x \in \partial\Omega,\ t \in R_+ \end{cases} \tag{1.5}$$

where $e_1^0(x) = u_0(x) - \bar{u}_0(x)$, $e_2^0(x) = v_0(x) - \bar{v}_0(x)$, and $\psi_1(e_1,e_2) = f(u,v) - f(\bar{u},\bar{v})$, $\psi_2(e_1,e_2) = g(u,v) - g(\bar{u},\bar{v})$.

We introduce the definition of equi-attractive in the large as following:

Definition 1 Solutions of impulsive system (1.5) are said to be equi-attractive in the large if for each $\epsilon > 0, \alpha > 0$ and $t_0 \in R^+$, there exist a number $Z = Z(t_0, \epsilon, \alpha) > 0$ such that $\|\mathbf{e}(0,t)\|_2 < \alpha$ implies $\|\mathbf{e}(x,t)\|_2 < \epsilon$ for $t \geq t_0 + Z$.

In [10] A. Khadra, X. Z. Liu and X. M. Shen study impulsive synchronization of system (1.1) in case $c = 0, \Omega = [0,L]$ and $f(u,v) = -uv^2 + a(1-u)$, $g(u,v) = uv^2 - (a+b)v$, obtain some sufficient conditions under which the synchronization error system is equi-atrractive in large.

II. Impulsive synchronization of coupled reaction-diffusion systems

For the convenience of depiction, the well-known Poincaré inequality is firstly introduced:

Lemma 3[11] If $u \in H_0^1(\Omega)$, then

$$\int_\Omega |u|^2 \mathrm{dx} \leq c_0 \int_\Omega |\nabla u|^2 \mathrm{dx} \tag{2.1}$$

where c_0 be called the Poincaré constant (if $\Omega = [0,L]$, then $c_0 = \frac{L}{\pi}$).

Theorem 4 Let

$$\begin{aligned} q_k^0 &= \max\{(1+q_k)^2, (1+p_k)^2\}, \\ a_0 &= \frac{1}{2}[a + d + \sqrt{c^2 + (a-d)^2}], \end{aligned}$$

if

$$(L_0 - a_0 c_0^2)\Delta_{k+1} + \ln q_k^0 \leq r_k, \quad \sum_{k=1}^{\infty} r_k = -\infty. \tag{2.2}$$

then system (1.5) is equi-attractive in the large.

Proof. Let us choose Lyapunov function

$$V(\mathbf{e}(x,t)) = \frac{1}{2}\int_\Omega (e_1^2(x,t) + e_2^2(x,t))dx.$$

For all $t \in (t_k, t_{k+1}],\ i = 1, 2, \ldots$, from (1.5), we obtain

$$\begin{aligned} D_t^+ V(\mathbf{e}) &= \int_\Omega \left(e_1 \frac{\partial e_1}{\partial t} + e_2 \frac{\partial e_2}{\partial t} \right) dx \\ &= \int_\Omega e_1(f(u,v) - f(\bar{u},\bar{v})) + a\int_\Omega e_1 \Delta e_1 \\ &\quad + \int_\Omega e_2(g(u,v) - g(\bar{u},\bar{v})) + c\int_\Omega e_2 \Delta e_1 \\ &\quad + d\int_\Omega e_2 \Delta e_2 \end{aligned} \tag{2.3}$$

According the assumptions (1.4), we have

$$\begin{aligned} & \int_\Omega e_1(f(u,v) - f(\bar{u},\bar{v})) + \int_\Omega e_2(g(u,v) - g(\bar{u},\bar{v})) \\ \leq\ & L_0 \|\mathbf{e}\|_2^2 \end{aligned}$$

According (2.3), we have

$$\begin{aligned} & D_t^+ V(\mathbf{e}) \\ \leq\ & L_0\|\mathbf{e}\|_2^2 - \int_\Omega \left(a|\nabla e_1|^2 + c\nabla e_1 \nabla e_2 + d|\nabla e_2|^2\right) \end{aligned} \tag{2.4}$$

For any $\varepsilon > 0$,

$$\begin{aligned} & -(a|\nabla e_1|^2 + c\nabla e_1\nabla e_2 + d|\nabla e_2|^2) \\ \leq\ & -(a - \frac{c\varepsilon}{2})|\nabla e_1|^2 - (d - \frac{c}{2\varepsilon})|\nabla e_2|^2. \end{aligned}$$

Noting that $c^2 - 4ad < 0$, we can take the ε satisfy the following condition

$$a - \frac{c\varepsilon}{2} = d - \frac{c}{2\varepsilon},$$

i.e.

$$\varepsilon_0 = \frac{a - d + \sqrt{c^2 + (a-d)^2}}{2}$$

let

$$a_0 = a - \frac{c}{2}\varepsilon_0 = \frac{a + d + \sqrt{c^2 + (a-d)^2}}{2}.$$

then, (2.4) becomes

$$D_t^+ V(\mathbf{e}) \leq L_0 \|\mathbf{e}\|_2^2 - a_0 \int_\Omega \left(|\nabla e_1|^2 |\nabla e_2|^2\right). \tag{2.5}$$

By the Poincare inequality

$$\int_\Omega (u_1^2 + u_2^2) \leq \int_\Omega (|\nabla u_1|^2 |\nabla u_2|^2), \ \mathbf{u} = (u_1, u_2) \in H_0^1(\Omega).$$

where c_0 is the Poincare constant (if $\Omega = [0, L]$, then $c_0 = \frac{L}{\pi}$)

From (2.5), we have

$$D_t^+ V(\mathbf{e}) \leq (L_0 - a_0 c_0^2) \|\mathbf{e}\|_2^2 \tag{2.6}$$

i.e.

$$V(\mathbf{e}(t_{k+1}, x)) = \exp(L_0 - a_0 c_0^2) V(\mathbf{e}(t_k^+, x)) \tag{2.7}$$

By the structure of the impulses given in system (1.5), from

$$e_1(t_k^+, x) - e_1(t_k, x) = q_k e_1(t_k, x),$$

$$e_2(t_k^+, x) - e_2(t_k, x) = p_k e_1(t_k, x),$$

we can obtain

$$e_1(t_k^+, x) = (1 + q_k) e_1(t_k, x),$$

$$e_2(t_k^+, x) = (1 + p_k) e_2(t_k, x).$$

i.e.

$$\begin{aligned} & V(\mathbf{e}(t_k^+, x)) \\ = \ & \frac{1}{2} \int_\Omega [(1 + q_k)^2 e_1^2(t_k, x) + (1 + p_k)^2 e_2^2(t_k, x)] dx \end{aligned} \tag{2.8}$$

let $q_k^0 = \max\{(1+q_k)^2, (1+p_k)^2\}$, then (2.7), (2.8) implies that

$$\begin{aligned} & V(\mathbf{e}(t_{k+1}, x)) \\ \leq \ & q_k^0 \exp[(L_0 - a_0 c_0^2)\Delta_{k+1}] V(\mathbf{e}(t_k, x)) \\ = \ & \exp[(L_0 - a_0 c_0^2)\Delta_{k+1} + \ln q_k^0] V(\mathbf{e}(t_k, x)). \end{aligned} \tag{2.9}$$

i.e.

$$\begin{aligned} & V(\mathbf{e}(t_{k+1}, x)) \\ \leq \ & \exp\left(\sum_{i=1}^{k} [(L_0 - a_0 c_0^2)\Delta_{i+1} + \ln q_i^0]\right) V(\mathbf{e}(t_1, x)) \\ \leq \ & \exp\left(\sum_{i=1}^{k} r_i\right) V(\mathbf{e}(t_1, x)). \end{aligned} \tag{2.10}$$

Therefore, from (2.2), we have

$$\lim_{k\to\infty} V(\mathbf{e}(t_k, x)) = 0 \tag{2.11}$$

III. Simulation results of impulsive synchronization

Consider the impulsive synchronization of the following coupled reaction-diffusion systems:

$$\begin{cases} \left.\begin{array}{l} \frac{\partial u_1}{\partial t} - a\Delta u_1 = 1 + u_1 + \cos v_1 \\ \frac{\partial v_1}{\partial t} - c\Delta u_1 - d\Delta v_1 = u_1 - \sin v_1 \end{array}\right\} x \in [0, L], t \in R^+ \\ u_1 = v_1 = 0, \ x = 0, L \\ u_1(x, 0) = u_{10}(x), v_1(x, 0) = v_{10}(x), \ x \in [0, L] \end{cases} \tag{3.1}$$

Here, system (3.1) is the driving system, and the driven system with impulsive control is given by

$$\begin{cases} \left.\begin{array}{l} \frac{\partial u_2}{\partial t} - a\Delta u_2 = 1 + u_2 + \cos v_2 \\ \frac{\partial v_2}{\partial t} - c\Delta u_2 - d\Delta v_2 = u_2 - \sin v_2 \end{array}\right\} t \neq t_k, x \in [0, L] \\ \left.\begin{array}{l} J u_2(x, t) = -q_k(u_1 - u_2) \\ J v_2(x, t) = -p_k(v_1 - v_2) \end{array}\right\} t = t_k, x \in [0, L] \\ u_2 = v_2 = 0, \quad x = 0, L, \ t \in R_+. \\ u_2(x, 0) = u_{20}(x), v_2(x, 0) = v_{20}(x), x \in [0, L] \end{cases} \tag{3.2}$$

Then, the impulsive synchronization error system is given by

$$\begin{cases} \left.\begin{array}{l} \frac{\partial e_1}{\partial t} = e_1 + \cos v_1 - \cos v_2 + a\Delta e_1 \\ \frac{\partial e_2}{\partial t} = e_1 - \sin v_1 + \sin v_2 + c\Delta e_1 + d\Delta e_2 \end{array}\right\} \begin{array}{l} t \neq t_k, \\ x \in [0, L] \end{array} \\ \left.\begin{array}{l} J e_1 = q_k e_1 \\ J e_2 = p_k e_2 \end{array}\right\} t = t_k, x \in [0, L] \\ e_1 = e_2 = 0, \quad x = 0, L, \ t \in R_+. \\ e_1(x, 0) = e_{10}(x), e_2(x, 0) = e_{20}(x), x \in [0, L] \end{cases} \tag{3.3}$$

where $e_1 = u_1 - u_2, e_2 = v_1 - v_2, e_{10}(x) = u_{10}(x) - u_{20}(x), e_{20}(x) = v_{10}(x) - v_{20}(x)$. Since $f(u_i, v_i) = 1 + u_i + \cos v_i, g(u_i, v_i) = u_i - \sin v_i, i = 1, 2$, let $L_0 = 1$ satisfying the condition (1.4). According to Theorem 4, we choose $r_k = -\ln\gamma$ $(\gamma > 1)$, obtain the following theorem.

Theorem 5 If $2\pi^2 - [a + d + \sqrt{c^2 + (a-d)^2}]L^2 > 0$, and

$$t_{k+1} - t_k < \frac{-2\pi^2 \ln(q_k^0 \gamma)}{2\pi^2 - [a + d + \sqrt{c^2 + (a-d)^2}]L^2}$$

where $\gamma > 1$, then the impulsive synchronization error system (3.3) is equi-attractive in the large.

In the following simulations, the initial conditions are given $(u_{10}(x), v_{10}(x)) = (1, 1)$ and $(u_{20}(x), v_{20}(x)) =$

$(1.5, 1.5)$, the impulses are equidistant from each other and separated by $\delta = 0.01$. We choose $L = 2$, $a = c = d = 1$, and $q_k = p_k = -\frac{1}{2}$, $\gamma = 1.5$ satisfying Theorem 5. Fig 1 and Fig 2 show the propagation of the solutions $u_1(t, x)$ and $v_1(t, x)$, respectively. Fig 3 shows the quick convergence of the error system to zeros in 0.5 s in which $\mathbf{e}(t, x) = (e_1, e_2)^{\mathrm{T}}$.

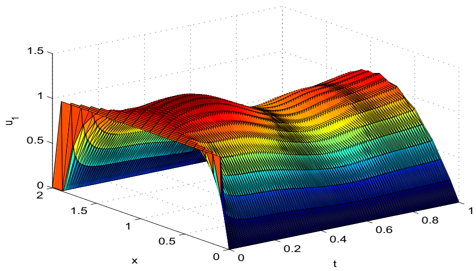

Figure 1: Propagation of $u_1(t, x)$ in system (3.1)

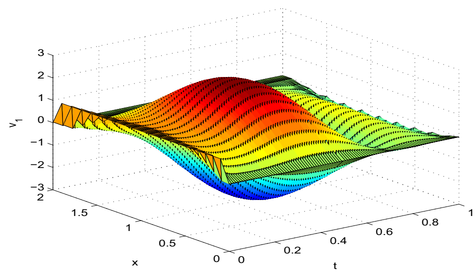

Figure 2: Propagation of $v_1(t, x)$ in system (3.1)

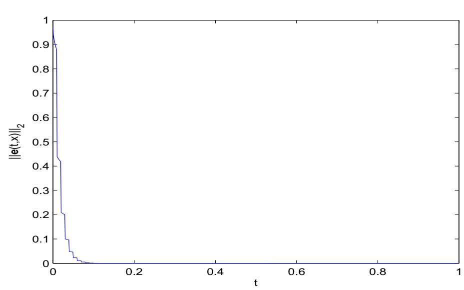

Figure 3: $\|\mathbf{e}(t, x)\|_2$ converging to zero in system (3.3)

IV. Application in information hiding

We are interested in applying the results from Section 2 and 3 to design an information hiding system with impulsive synchronization. The proposed system contains two partial differential systems (u_1, v_1) and (u_2, v_2), not necessarily identical, which are used to mask-modulate at the transmitter end and unmask the information signal $m(t, x)$ at the receiver end, respectively. The masking-modulating process of $m(t, x)$ is done through the operation: $h(t, x) = H(m, u_1, v_1)$. $s(t, x)$ is the transmitted signal; it consists of a sequence of time frames T. Each time frame is divided

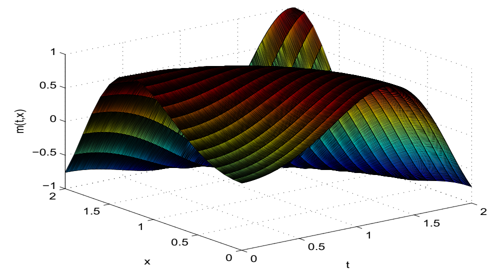

Figure 4: Original signal $m(t, x) = \sin(t^2 + x^2)$

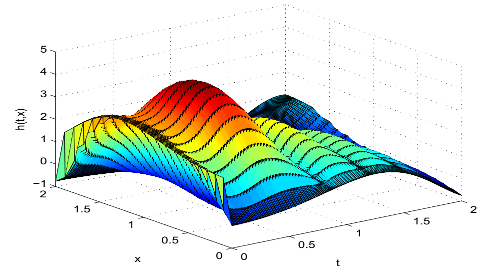

Figure 5: Encrypted signal $h(t, x)$

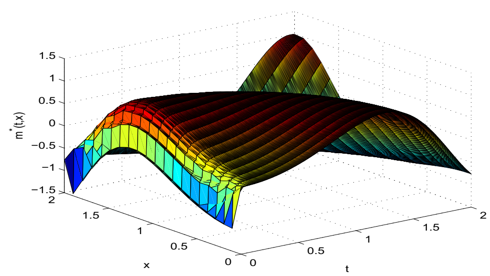

Figure 6: Recovered signal $m^*(t, x)$

into two regions: Synchronizing-impulses region of length Q, where the impulses are loaded, and the encryption region of length $T - Q$, where $h(t, x)$ is loaded. The two regions are combined at the transmitter and the sent across a public channel to the receiver. At the receiver, each time frame of $s(t, x)$ is decomposed into the encrypted message $h(t, x)$ and the synchronizing impulses. At this point, $h(t, x)$ is used to drive the system (u_2, v_2), whereas the impulses are used to impulsively synchronize (u_2, v_2) with (u_1, v_1). When synchronization is achieved, we have $(u_2, v_2) \approx (u_1, v_1)$. Thus, the decryption process becomes feasible and (u_2, v_2) may be used to recover the original information in the following way

$$m(t, x) \approx m^*(t, x) = H^{-1}(h, u_2, v_2).$$

As an example of the above scheme, we shall try to hide the information signals given by $m(t, x) = \sin(t^2 + x^2)$. This will be done use the two coupled reaction-diffusion systems (3.1) and (3.2). The masking-modulating process of $m(t,x)$ is done through $h(t,x) = m + u_1 + v_1$. The relevant parameters of systems and scheme of impulsive synchronization are chose as in Section 3. The original signal $m(t,x)$ and the encrypted signal $h(t,x)$ are shown in Fig 4 and 5, respectively. Fig 6 shows the recovered signal $m^*(t,x)$. The simulation results show the information hiding scheme is effective.

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