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A Methodology for Determining the Non-Existence of Common Quadratic Lyapunov Functions for Pairs of Stable Systems

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***Abstract*—The existence of a common quadratic Lyapunov function (CQLF) for a switched linear system guarantees its global asymptotic stability. Although the progress in finding conditions for existence/non-existence of a CQLF has been significant in the last years, especially in switched linear systems with N subsystems of second order or two non-arbitrary subsystems of order n, the general case of N systems of order n still remains open. In this paper, based on a sufficient condition for the non-existence of a CQLF for a pair of general subsystems of order n obtained from a lemma by Shorten et al., a new method for determining the non-existence of a CQLF, using Particle Swarm Optimization, is designed. A example illustrating the proposed method is introduced towards the end of the paper.**

***Index Terms*—Common quadratic Lyapunov function; Particle Swarm Optimization; stability of switched systems;**

I. Introduction

Linear dynamical systems that evolve by switching among several evolution matrices through a commutation rule are called switched linear systems (SLS). In continuos time, such systems can be defined as follows ([1]):

$$\dot{x}(t) = A_{\sigma(t)} x\left(t\right), \quad A_{\sigma(t)} \in \mathrm{A} := \left\{A_1, A_2, ..., A_N\right\} \tag{1}$$

with $A_i \in \Re^{n\times n}$, $i=1,2,...,N$ Hurwitz matrices. The matrix $A_{\sigma(t)}$ takes the values $A_i \in \mathrm{A}$ in time intervals given by the switching rule $\sigma\left(t\right) \in \Re^{+}$, i.e. $A_{\sigma(t)} = A\left(t\right)$ is constant between consecutive switching times ([1]). Mathematical models of SLS are more appropriate to describe the behavior of many processes and multi model/controller systems, that appears in real field applications in engineering and others areas ([2]).

Currently, the main effort in the SLS field is focused on studying the dynamical performance, controller design, and stability analysis. The latter is a critical issue as far as operator safety and hardware integrity are concerned, which is mainly affected by system failures. Since the subsystems stability does not guarantee the stability of the whole switched system, because some $\sigma\left(t\right)$ can make unstable the system, it is necessary to analyze the stability of the whole switched system. In the case of SLS, an important stability issue is related to the existence of common Lyapunov functions. More particularly, the existence of a common quadratic Lyapunov function (CQLF) for a SLS guarantees its global stability ([2]).

In general, studies to establish CQLF existence/non-existence conditions begin analyzing pair of matrices and then try to apply these results to a set of N matrices. In [3] the case when the Lie bracket is equal to zero is studied, concluding that the commutativity of two matrices is a sufficient condition for sharing a CQLF. An algorithm for finding such CQLF is designed, which is also successfully applied to N matrices that commute pairwise. Later in [4] the scope of such algorithm is expanded to the case when the Lie bracket can be expressed as a convex combination with constrained coefficients. The most important result related with a pair of second order systems is the reported in [5]. There, a solution based on the stability analysis of the convex linear combination (CLC) of two matrices is shown. Then, the case of N systems is faced using the Helly's Theorem, achieving so a complete analytical solution for a finite set of second order systems.

Several other approaches have been used to solve the CQLF problem (e.g. [1], [2], [6], [7]), but most of the approaches uses strong constrains on the order/number of systems or have to satisfy special properties, and the determination of general CQLF existence/non-existence conditions is not completely solved problem. Thus in this paper, a methodology for determining the of non-existence of a CQLF for a pair of n-order systems is proposed, from a necessary existence condition of a CQLF presented in [7] and based on the global optimization technique Particle Swarm Optimization (PSO).

II. Problem formulation and basic concepts

In what follows, for a pair of matrices $A_1, A_2 \in \Re^{n\times n}$, the matrix $\sigma_\alpha\left[A_1, A_2\right] = \alpha A_1 + \left(1-\alpha\right) A_2, \forall \alpha \in \left[0,1\right]$, denotes the convex combination of A_1, A_2, commonly referred as pencil of A_1, A_2 and said Hurwitz if its eigenvalues lies in the open left half-plane of the complex plane $\forall \alpha \in \left[0,1\right]$ ([5]).

Let us consider the continuous SLS given by (1), and let $V\left(x\right) > 0$ be a quadratic Lyapunov function candidate as

$$V\left(x\right) = x^T P x, \quad P > 0, \quad P \in \Re^{n\times n}. \tag{2}$$

Then, if there exists a matrix $P > 0$ satisfying

$$PA_i + A_i^T P = -Q_i < 0, \quad \forall A_i \in \mathrm{A}, \tag{3}$$

the function $V\left(x\right)$ is a CQLF for all subsystems of the form

$$\Sigma_{A_i} : \dot{x}\left(t\right) = A_i x\left(t\right), \quad i = 1,2,...,N, \quad \forall A_i \in \mathrm{A} \tag{4}$$

and its existence guarantees uniform asymptotic stability of the SLS (1) ([1], [2]). In other words, the fact that P satisfies (3) implies that the time derivative of $V\left(x\right)$ is negative definite along any system trajectory, i.e.

$$\dot{V}\left(x\right) = x^T \left(PA_i + A_i^T P\right) x < 0, \quad \forall A_i \in \mathrm{A}, \tag{5}$$

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i.e., all Lyapunov equations are solved simultaneously with P.

For the sake of completeness, lemmas and theorems used in the design of the proposed methodology are presented below.

Lemma 1 ([5]): Let us consider the systems $\Sigma_A : \dot{x} = Ax$ and $\Sigma_{A^{-1}} : \dot{x} = A^{-1}x$ where $A \in \Re^{n\times n}$ is Hurwitz. Then, any quadratic Lyapunov function for Σ_A is also a quadratic Lyapunov function for $\Sigma_{A^{-1}}$.

Theorem 1 ([5]): Let us consider the system (1) with $x \in \Re^2$ and $N = 2$. The following statements are equivalents:

(1) There exists a CQLF for (1) with $\mathrm{A} = \{A_1, A_2\}$.
(2) The pencils $\sigma_\alpha [A_1, A_2]$ and $\sigma_\alpha \left[A_1, A_2^{-1}\right]$ are Hurwitz.
(3) The products A_1A_2 and $A_1A_2^{-1}$ do not have real negative eigenvalues.

Lemma 2 ([7]): If the stable LTI systems $\dot{x} = A_1x$ and $\dot{x} = A_2x$, with $A_1, A_2 \in \Re^{n\times n}$, have a CQLF, then the pencils $\sigma_\alpha [A_1, A_2]$ and $\sigma_\alpha \left[A_1^{-1}, A_2\right]$ are non-singulars. Equivalently, $A_1^{-1}A_2$ and A_1A_2 do not have real negative eigenvalues.

III. The PSO-based methodology

We present the development of the PSO-based methodology proposed for determining the non-existence of CQLF.

A. The PSO algorithm

PSO ([8], [9]) is a global optimization technique from Evolutionary Computation (EC) that computationally emulates the social behavior of communities as fish schools, bird flocks, or even a crowd. Thus, PSO belongs to the class of modern heuristic techniques based on population (a particle swarm), in which each particle is a potential solution for problem to be solved, and has the ability to move in a multidimensional search space with velocity and position updated in its standard version PSOiw (PSO inertia weighted [9]) by

$$\begin{aligned} v_{i,d}(k+1) = \omega v_{i,d}(k) + r_{1,d}(k)\, c_1 \left(p_{i,d}(k) - x_{i,d}(k)\right) \\ + r_{2,d}(k)\, c_2 \left(g_d(k) - x_{i,d}(k)\right), \end{aligned} \quad (6)$$

$$x_{i,d}(k+1) = x_{i,d}(k) + v_{i,d}(k+1), \quad (7)$$

$$\mathbf{x}_i = [x_{i,1}, ..., x_{i,D}], \; i \in \{1, ..., s\}, \; d \in \{1, ..., D\},$$

where i, d, k denote the particle, dimension and iteration counter respectively, s is the swarm size, D is the dimension of the function fitness f (given by the number of unknowns), the random variables $r_1, r_2 \sim U[0,1]$ represent the stochastic nature of any swarm population, c_1, c_2 are the cognitive and social acceleration coefficients that determine the individual and group experience influence respectively. The parameter $\omega \in [0,1]$ is the inertia weight, and basically contributes to the convergence of the particles and the stability analysis. The best individual position $\mathbf{p}_i(k) = [p_{i,1}(k), ..., p_{i,D}(k)]$ and the best global position $\mathbf{g}(k) = [g_1(k), ..., g_D(k)]$ are updated by

$$\mathbf{p}_i(k) = \begin{cases} \mathbf{p}_i(k-1), & \text{if } f(\mathbf{p}_i(k-1)) \le f(\mathbf{x}_i(k)) \\ \mathbf{x}_i(k), & \text{if } f(\mathbf{p}_i(k-1)) > f(\mathbf{x}_i(k)) \end{cases} \quad (8)$$

$$\mathbf{g}(k) = \text{argmin}\{f(\mathbf{p}_1(k)), ..., f(\mathbf{p}_s(k))\}. \quad (9)$$

Relevant applications of PSO may be found in [9]-[11].

B. The PSO-based methodology: Step-by-step

The method of determining the non-existence of a CQLF is based on Lemma 2. As such, will be interested in determining if for a given pair of matrices A_1, A_2 some of the pencils

$$\begin{aligned} M_1(\alpha) = \sigma_\alpha [A_1, A_2], M_2(\alpha) = \sigma_\alpha \left[A_1^{-1}, A_2\right], \\ M_3(\alpha) = \sigma_\alpha \left[A_1, A_2^{-1}\right], M_4(\alpha) = \sigma_\alpha \left[A_1^{-1}, A_2^{-1}\right] \end{aligned} \quad (10)$$

are non singular. For understanding better the design process of the PSO-based methodology, such a process will be divided in the following steps:

1) Fitness function design.
2) Coding of potential solutions.
3) Analysis of the optimization process outputs.

In the next subsections, this steps will be explained in detail.

1) Fitness function design: As important as the parameter settings of PSO is the definition of the associated fitness function. PSO has the inherent advantage of heuristic methods of using fitness functions that are not too restrictive as in the case of deterministic techniques, being able to manage non-differentiable functions, including non linearities, discontinuities and constraints ([11]). However, it is known that by using a non-suitable fitness function to measure the goodness of the particles evaluated, a poor performance may be encountered, regardless of the technique itself or its parameter settings.

Thus, it is necessary to design a suitable fitness function to be optimized using PSO, for the problem of non-existence of a CQLF for a pair of stable matrices, which is directly related to Lemma 2. According to Lemma 2, the non-singularity of $\sigma_\alpha \left[A_1, A_2^{\pm 1}\right]$ is a consequence of the existence of a CQLF for $\mathrm{A}=\{A_1, A_2\}$, that is to say, the non-singularity of these two pencils is a necessary condition for the elements in A to share a CQLF. However, given that $\sigma_\alpha \left[A_1, A_2^{\pm 1}\right] \subset \left\{CLC\left[A_i^{\pm 1}, A_j^{\pm 1}\right]\right\}_{i,j=1,2}$, then by using Lemma 1 follows that is only necessary to evaluate the non-singularity of

$$CLC\left[\left\{A_{1,2}^{\pm 1}\right\}\right] = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_1^{-1} + \alpha_4 A_2^{-1} \quad (11)$$

subject to

$$\textstyle\sum_{k=1}^{4} \alpha_k = 1, \; \alpha_{1,2,3,4} > 0. \quad (12)$$

Note that the singularity of (11) requires a

$$\Lambda = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \quad (13)$$

producing a non-Hurwitz matrix

$$W(\Lambda) = CLC\left[A_1, A_2, A_1^{-1}, A_2^{-1}\right], \quad (14)$$

i.e., $\{\text{Real}[\text{eig}(-W(\Lambda))] \le 0\} \neq \emptyset$. From the optimization viewpoint, this last expression is appropriate for designing a function susceptible to be minimized, resulting in

$$f(\Lambda) = \min\{\text{Real}[\text{eig}(-W(\Lambda))]\} \quad (15)$$

subject to (12), where $\text{eig}(\bullet)$ calculates matrix eigenvalues.

Remark 1: It is interesting to observe from Theorem 1 that for the case of n=2, the non-singularity of W ensures the existence of a CQLF. However, from the same Theorem 1 it is sufficient to check the eigenvalues of A_1A_2 and $A_1A_2^{-1}$

to assess non-singularity of the two pencil, because the non-singularity of $\sigma_\alpha\left[A_1, A_2^{-1}\right]$ implies the non-singularity of $\sigma_\alpha\left[A_1^{-1}, A_2\right]$ and vice versa (see Proof of Theorem 1 in [5]). However, this is not generally true for the case of $n>2$ (as shown below in the Example). Then, the method proposed is not that useful when $n=2$ as in the case of $n>2$.

2) Coding of potential solutions: It is seen that its fits to PSO technique is direct, by relating Λ with the particles to evolve. Remembering that the position of an arbitrary particle is $\mathbf{x}=[x_1,...,x_D]$, then it is desired to find a function

$$\nu\left(\mathbf{x}\right)=\Lambda, \tag{16}$$

such that $\nu\colon R^D \to R^4$ relates (13) with (16), satisfying also (12). Since Λ and $\mathbf{x}$ represent the same element through ν, it follows $f\left(\mathbf{x}\right) \doteq f\left(\Lambda\right)$. Then the fitness function is defined as

$$f\left(\Lambda\right)=\min\left\{\text{Real}\left[\text{eig}\left(-W\left(\nu\left(\mathbf{x}\right)\right)\right)\right]\right\}. \tag{17}$$

A specific fitness function is determined depending on how ν is defined. The following choice for ν is proposed:

1. Let us consider $D=3$, and

$$x_{1,2,3}\in\left[0,1\right]. \tag{18}$$

2. To calculate $\widetilde{\alpha}_k$ of the form

$$\widetilde{\alpha}_k=\begin{cases} x_k & k=1,2,3. \\ 1-\text{mod}\left(\sum_{h=1}^{3}\tilde{\alpha}_h,1\right) & k=4. \end{cases} \tag{19}$$

where $\text{mod}\left(a,b\right)$ is the modulus after $\frac{a}{b}$ division.

3. Let us define $K=\sum_{k=1}^{4}\tilde{\alpha}_k$, and calculate α_k as

$$\alpha_k=\frac{\tilde{\alpha}_k}{K},\ k=1,...,4. \tag{20}$$

It can be seen that (20) satisfy (12). Then, the set formed by equations (16)-(20) constitutes the fitness function to be used.

3) Analysis of the optimization process outputs: The final output may be: i) a positive, or ii) a non-positive real number.

By definition of the fitness function itself (see Subsection III-B1), the only case in which the output is conclusive occurs when is of kind ii). Thus, the fitness function only delivers useful results by obtaining $f\left(\Lambda\right)\leq 0$, i.e., when crossing a threshold in zero that allows to deduce the non-existence of a CQLF by a counter example of its existence, because $f\left(\Lambda\right)\leq 0$ means that some Λ produces a non-Hurwitz W (and so singular at least for two Λ), and this violates the necessary CQLF existence condition given by Lemma 2. In the case of a positive output just it can be said that is a hope of the CQLF existence, and this is because Lemma 2 only offers a necessary condition for the existence of a CQLF, but not sufficient.

IV. Experimental results

Performance evaluating of the proposed methodology involved programming in Matlab, using the PSO ToolBox developed by Jagatpreet Singh ([12]). It was necessary an auxiliary file to define the fitness function, and a modification to the file PSO.m to define the initial positions: i) random (uniform distribution), and ii) predefined, starting the optimization process only with equally weighted pairwise CLCs.

All tests were made using PSOiw, with the following parameter configuration: inertia weight $\omega\left(k\right)$ decreasing linearly with respect to k, from $\omega\left(0\right)=0.9$ to $\omega\left(k_{max}\right)=0.4$, together with $c_1=c_2=2$ as recommends [9]. The population size was chosen as $s=8$ (usually recommended $s\geq 2D=6$). As termination criteria were chosen two working simultaneously: i) crossing the threshold $f\left(\Lambda\right)=0$, and ii) achieving the maximum number of iterations ($k>k_{max}=80$, by default). Finally, the progress of the optimization process is shown every $0.1*k_{max}$ iterations, where $fGBest$ is the best fitness value found at $Iteration$.

A. Example: Two matrices in $\Re^{10\times 10}$ (no CQLF)

Two randomly generated stable matrices $A_1=\left[A_{1,1},A_{1,2}\right]$ and $A_2=\left[A_{2,1},A_{2,2}\right]$ are chosen, where

$$A_{1,1}=\begin{bmatrix} 5.09 & -0.66 & 12.90 & -6.27 & 8.70 \\ 7.99 & -10.78 & -1.22 & 6.55 & -3.03 \\ -8.31 & -5.26 & -13.77 & 9.82 & -9.67 \\ 7.31 & -11.88 & -12.81 & -3.34 & 9.62 \\ -7.70 & -5.61 & 5.53 & -15.08 & -14.67 \\ -19.44 & 6.21 & -6.47 & 5.29 & 10.42 \\ 4.44 & -9.44 & -0.45 & -6.07 & -2.53 \\ 6.21 & 18.29 & -0.59 & -10.57 & 0.40 \\ 2.79 & 7.75 & 8.06 & 1.26 & -6.30 \\ -4.26 & 4.73 & -7.53 & 14.70 & -25.50 \end{bmatrix},$$

$$A_{1,2}=\begin{bmatrix} 18.43 & -7.91 & 12.29 & -2.84 & -15.16 \\ 2.59 & 4.13 & -12.61 & -2.03 & -6.86 \\ -5.32 & -5.38 & 0.47 & 2.55 & 13.97 \\ -0.56 & -2.31 & 10.73 & 3.65 & -7.31 \\ -12.76 & -2.86 & 13.49 & -4.26 & 6.74 \\ -11.98 & -4.62 & 9.32 & 4.90 & -5.55 \\ 5.72 & -3.87 & -1.55 & 0.58 & 1.48 \\ 5.45 & 10.39 & -17.25 & 17.59 & 4.47 \\ 5.05 & 2.88 & -33.39 & -6.90 & -6.81 \\ -0.74 & -21.09 & 10.22 & 13.70 & -9.34 \end{bmatrix},$$

$$A_{2,1}=\begin{bmatrix} -19.59 & -3.57 & -1.88 & 1.71 & 15.55 \\ 10.76 & -2.89 & -6.64 & -15.90 & -30.19 \\ 3.97 & -16.90 & 4.27 & 10.33 & -21.21 \\ -6.41 & 12.71 & -3.79 & -15.86 & 7.16 \\ -1.35 & 11.86 & -0.33 & -1.94 & -17.75 \\ 2.94 & 10.42 & -0.30 & 1.26 & 4.24 \\ 7.46 & 2.83 & 8.09 & 12.83 & -5.60 \\ 10.08 & 11.95 & -4.25 & -4.36 & 14.34 \\ 16.66 & 0.27 & -4.21 & -0.77 & -10.77 \\ -2.22 & -5.57 & 0.27 & 7.65 & 6.61 \end{bmatrix},$$

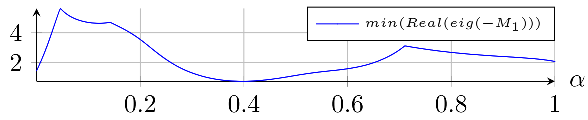
$$A_{2,2}=\begin{bmatrix} 2.50 & 11.11 & -6.92 & -8.80 & 5.78 \\ -3.80 & -3.73 & -15.35 & 14.09 & 8.18 \\ 6.85 & -7.85 & -6.58 & 11.58 & -0.24 \\ 0.86 & -16.90 & -6.96 & -10.15 & 1.14 \\ 5.21 & -21.96 & -18.02 & -15.64 & -9.00 \\ -14.65 & 11.06 & 10.05 & 7.43 & -2.62 \\ -9.52 & -10.78 & 9.38 & 20.24 & -1.99 \\ 6.60 & -6.49 & -9.00 & 1.76 & 28.66 \\ -7.89 & -16.14 & -8.59 & -12.91 & -5.27 \\ -0.19 & -2.97 & -11.88 & 1.12 & -7.03 \end{bmatrix},$$

and no products $A_1^{\pm 1}A_2^{\pm 1}$ have real negative eigenvalues.

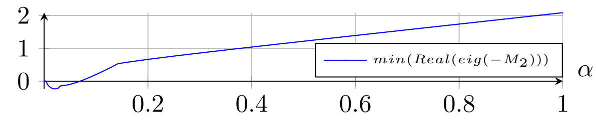
Fig. 1 presents the singularity analysis of the pencils (10) generated, and it can be concluded that M_2 is non-Hurwitz for $\alpha\approx\left[0.004, 0.068\right]$. Based on this information it follows analytically that $\text{A}=\left\{A_1,A_2\right\}$ do not share a CQLF. Table I shows the results of applying the proposed methodology, corroborating the non-existence of a CQLF by obtaining a non-positive output. It is seen that in this case the results are also not very sensitive to the population initialization employed.

V. Conclusions

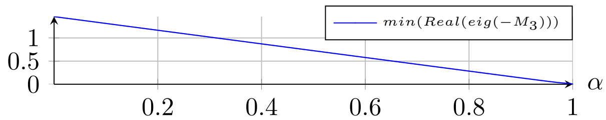
In this paper, a PSO-based methodology for determining the non-existence of a CQLF for a pair of stable systems it was reported. Results obtained show that PSO is successfully applied in the search for solutions to the CQLF problem. The designed methodology was tested for different order



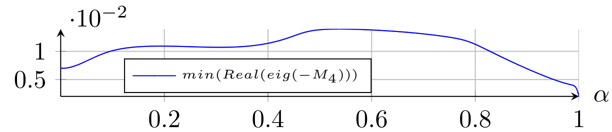
(a) Analysis of $M_1(\alpha)$



(b) Analysis of $M_2(\alpha)$



(c) Analysis of $M_3(\alpha)$



(d) Analysis of $M_4(\alpha)$

Fig. 1. Graphical singularity analysis of the four pencils (10) generated from matrices A_1, A_2 of the Example, as a function of parameter α.

of matrices, exhibiting in general a good performance that slightly deteriorates when increasing the dimensionality of the problem, affecting shortly the convergence speed.

TABLE I
SAMPLE OF THE PSO-BASED METHODOLOGY APPLICATION TO THE EXAMPLE. FINAL OUTPUTS ARE UNDERLINED.

Randomized initial positions		**Predefined initial positions**	
Iteration	**fGBest**	**Iteration**	**fGBest**
8	0.1209	8	0.1303
16	0.1209	16	0.1303
17	<u>-0.24051</u>	24	0.010316
		27	<u>-0.019013</u>
Final eigenvalues: {-0.21±1.07i, -1.48, **0.24**, -0.76, -0.29±0.69i, -0.29±0.22i, -0.96}		Final eigenvalues: {-0.21±1.38i, -0.33±0.90i, -0.45, -1.36, -0.97±0.24i, **0.02**, -0.35}	
Elapsed time: 0.1613 s.		**Elapsed time:** 0.1956 s.	

When the final optimization process output is a non-positive number, the methodology assures with certainty 1 that the pair of matrices under analysis do not share a CQLF. However, since it is the case of verifying a sufficient condition for the non-existence of a CQLF, when the output of the optimization process is a positive number the proposed methodology is not able to conclusively guarantee whether exists or not a CQLF for matrices under analysis. Nevertheless, the proposed methodology is an important tool, eliminating the uncertainty of whether a pair of stable matrices shares a CQLF or not when finds evidence of non-existence, and the experience showed that it is able of doing so in a highly successful way.

By using the graphical method may be necessary to use a number of samples relatively large when the order of the matrices is increased. However, when the proposed methodology was applied, good results were observed using the same configuration, even in the case of 10-th order matrices. In any case, the time consumed by the methodology is minimum, involving an average of 0.1827 seconds to find a solution by evaluating about 180 times the fitness function in 23 iterations. This information was extracted from data obtained in 10 runs. PSOiw exhibited good results in this particular case regarding the ability to find a feasible solution with fast convergence speed. Eventually better results could be obtained if a newer particle swarm optimizer (e.g. [9], [11]) is used.

Finally, it is important to note that the analysis of pairs of systems is a preliminary step to analyze the general case, and thus the main idea of the methodology proposed may be extended to the case of N systems broadening its scope. Investigations on these topics are currently underway.

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REFERENCES

[1] R.N. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems", *SIAM Review*, 49(4), 2007, pp. 545-592.
[2] H. Lin, and P.J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results", *IEEE Transactions on Automatic Control*, 54(2), 2009, pp. 308-322.
[3] K.S. Narendra, and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-matrices", *IEEE Transactions on Automatic Control*, 39(12), 1994, pp. 2469-2471.
[4] Y.H. Zhu, D.Z. Cheng, and H.S. Qin, "Constructing common quadratic Lyapunov functions for a class of stable matrices", *Acta Automatica Sinica*, 33(2), 2007, pp. 202-204.
[5] R.N. Shorten, and K.S. Narendra, "Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for a finite number of stable second order linear time-invariant systems", *Int. J. Adapt. Control and Signal Process*, 16, 2002, pp. 709-728.
[6] T.J. Laffey, and H. Smigoc, "Common Lyapunov solutions for two matrices whose difference has rank one", *Linear Algebra and its Applications*, 431(1-2), 2009, pp. 228-240.
[7] R.N. Shorten, O. Mason, F.O. Cairbre, and P. Curran, "A unifying framework for the SISO circle criterion and other quadratic stability criteria", *International Journal of Control*, 77(1), 2004, pp. 1-8.
[8] R.C. Eberhart, and J.F. Kennedy, "Particle swarm optimization", in *Proceedings of IEEE International Conference on Neural Networks (ICNN)*, Piscataway, NJ, 4, 1995, pp. 1942-1948.
[9] Y. Del Valle, G.K. Venayagamoorthy, S. Mohagheghi, J.C. Hernandez, and R.G. Harley, "Particle swarm optimization: Basic concepts, variants and applications in power systems", *IEEE Transactions on Evolutionary Computation*, 12(2), 2008, pp. 171-195.
[10] C.H. Hsu, W.J. Shyr, and K.H. Kuo, "Optimizing multiple interference cancellations of linear phase array based on particle swarm optimization", *Journal of Information Hiding and Multimedia Signal Processing*, 1,(4), 2010, pp. 292-300.
[11] C.L. Sun, J.C. Zeng, and J.S. Pan, "An improved vector particle swarm optimization for constrained optimization problems", *Information Sciences*, 181(6), 2011, pp. 1153-1163.
[12] J. Singh, 2003, "The PSO ToolBox", http://sourceforge.net/projects/psotoolbox [accessed 2011-03-18].